Ph 236 – Homework 5
Due: Wednesday, November 7, 2012

1. Circumference of a circle. [18 points]
We've provided several descriptions in class of what information the Riemann tensor contains. In this exercise you will work through one more. For this problem, you should consider an n-dimensional space of Euclidean signature.

Consider a point $\mathcal{O}$. Given two orthogonal unit vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ at $\mathcal{O}$, we may construct a *circle* of radius ρ around $\mathcal{O}$ as follows. First, consider the sequence of unit vectors $\boldsymbol{v}(0,n)$ at $\mathcal{O}$ given by

$$\boldsymbol{v}(0,n) = \cos n\, \boldsymbol{a} + \sin n\, \boldsymbol{b}, \tag{1}$$

where $0 \le n < 2\pi$ is an angle. Now for each n, let us project a geodesic from $\mathcal{O}$ parameterized by λ according to the usual rule $D\boldsymbol{v}/\partial\lambda = 0$, and describe this family of geodesics by $\mathcal{P}(\lambda,n)$. The sequence of points at $\lambda = \rho$, i.e. at distance ρ from $\mathcal{O}$, we describe as the circle $\mathcal{C}(n)$. In this problem, we will consider the circumference of the circle.

(a) Define the vector $\boldsymbol{\xi} = \partial\mathcal{P}/\partial n$ and the scalar $\psi = \boldsymbol{\xi}\cdot\boldsymbol{\xi}$. Show that at $\lambda = 0$ we have the derivatives of $\boldsymbol{\xi}$:

$$\boldsymbol{\xi}(0,n) = 0, \quad \frac{D\boldsymbol{\xi}}{d\lambda}(0,n) = \boldsymbol{v}(0,n+\pi/2), \quad \frac{D^2\boldsymbol{\xi}}{d\lambda^2}(0,n) = 0, \tag{2}$$

and

$$\frac{D^3\boldsymbol{\xi}}{d\lambda^3}(0,n) = -\boldsymbol{Riemann}\left(\_, \boldsymbol{v}(0,n), \boldsymbol{v}(0,n+\pi/2), \boldsymbol{v}(0,n)\right). \tag{3}$$

(b) Now define the scalar $\psi = \boldsymbol{\xi}\cdot\boldsymbol{\xi}$. Find its derivatives at $\lambda = 0$ up through $\partial^4\psi/\partial\lambda^4$. One or more of your expressions will contain $\boldsymbol{Riemann}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{a},\boldsymbol{b})$.

(c) Find an expression, valid through order λ^3, for the norm $|\boldsymbol{\xi}|$. By relating this to the arc length along the circle, show that the circumference of the circle is

$$C = 2\pi\rho - \frac{\pi}{3}[\boldsymbol{Riemann}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{a},\boldsymbol{b})]\rho^3 + \text{higher order terms}. \tag{4}$$

Thus we see that the Riemann tensor describes the lowest-order deviation from the Euclidean formula $C = 2\pi\rho$.

2. Geodetic precession. [18 points]
Consider an observer with 4-velocity $\boldsymbol{u}$ who parallel-transports a unit vector $\boldsymbol{l}$ with $\boldsymbol{l}\cdot\boldsymbol{u} = 0$ along their trajectory. Imagine $\boldsymbol{l}$ to be the direction given by a gyroscope. Suppose also that our observer is in a spacetime that is dominated by nonrelativistic sources, so that the line element is

$$ds^2 = -(1+2\Phi)\,dt^2 + (1-2\Phi)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] \tag{5}$$

with $\Phi \sim \mathcal{O}(\epsilon)$, and that the observer moves relative to the coordinate system with $u^i \sim \mathcal{O}(\epsilon^{1/2})$. The treatment of 3-velocity as being of order the square root of the potential is of course appropriate for the motions of virialized systems – e.g. the Solar System – where kinetic and potential energies are of the same order of magnitude. The trajectory of the observer can be parameterized by $x^i(t)$, where t is the coordinate time; in this problem, we will use overdots to denote d/dt (not $d/d\tau$).

The system is assumed to evolve on a dynamical timescale $t_{\rm dyn}$, and hence has a length scale of order $t_{\rm dyn}\epsilon^{1/2}$. [In the Solar System, $t_{\rm dyn} \sim 1$ year, $\epsilon \sim 10^{-8}$, and the length scale is $t_{\rm dyn}\epsilon^{1/2} \sim 1$ hour, which is a few AU in relativistic units where $c = 1$.]

(a) Show that to order ϵ

$$l^0 = \dot{x}^i l^i. \tag{6}$$

(b) By careful counting of the powers of ϵ and $t_{\rm dyn}$, show that to order $\epsilon/t_{\rm dyn}$:

$$\frac{dl^i}{dt} = -2\dot{x}^j l^j \Phi_{,i} + l^i \dot{\Phi} + \dot{x}^i l^k \Phi_{,k} + \dot{x}^l l^i \Phi_{,l}. \tag{7}$$

[In carrying out this calculation, you should *not* need to explicitly compute $dt/d\tau$.]

We now wish to consider the long-term evolution of the direction provided by the gyroscope, $\boldsymbol{l}$ – i.e. we will average the result from part (b) over many dynamical times. We will assume that the orbit of our observer is virialized, i.e. that the velocities and potential may fluctuate but there is no secular (long term) trend: $\langle\dot{\Phi}\rangle = 0$, etc.

(c) Show that under the case of virialization that

$$\langle \dot{x}^i \ddot{x}^j \rangle \tag{8}$$

is antisymmetric. Thus its entire information content is contained in

$$B^k = \epsilon^{ijk} \langle \dot{x}^i \ddot{x}^j \rangle \quad \text{or} \quad {}^{(3)}\boldsymbol{B} = \langle {}^{(3)}\dot{\boldsymbol{x}} \times {}^{(3)}\ddot{\boldsymbol{x}} \rangle. \tag{9}$$

For the Newtonian equations of motion, use virialization to express $\langle \dot{x}^i \Phi_{,j} \rangle$ in terms of B^k; your expression should be accurate of order $\epsilon/t_{\rm dyn}$.

(d) Show that then the rate of change of direction of the gyroscope is

$$\left\langle \frac{dl^i}{dt} \right\rangle = \frac{3}{2} \epsilon^{ijk} B^j l^k \quad \text{or} \quad \langle {}^{(3)}\dot{\boldsymbol{l}} \rangle = \frac{3}{2} {}^{(3)}\boldsymbol{B} \times {}^{(3)}\boldsymbol{l}. \tag{10}$$

Thus a gyroscope carried by the observer appears to precess at a rate given by $\boldsymbol{B}$ (in both magnitude and in direction, i.e. axis of precession) relative to the outside Universe.

(e) Evaluate $\boldsymbol{B}$ for an observer on a circular orbit of radius R around a central mass M. In angular units per year (e.g. arcsec/yr, degrees/yr ...) what is this for a low-orbiting satellite around the Earth? [Yes, this has actually been measured!]