Ph 236 – Homework 4
Due: Wednesday, October 31, 2012

1. Gravitational "fields" and the equivalence principle. [18 points]

Consider a spacetime whose metric is given by

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}, \tag{1}$$

where $h_{\alpha\beta}$ are perturbations $\ll 1$. This is a small perturbation on Minkowski space, and hence we expect (correctly) that it can be used to describe nonrelativistic gravitational fields. We denote its time-time and time-space components by the shorthand

$$h_{00} = -2\Phi \quad \text{and} \quad h_{0i} = A_i, \tag{2}$$

thus defining the 3-scalar Φ and 3-vector A_i (i.e. these act like scalars and vectors in $\mathbb{R}^3$, but not in the full spacetime).

(a) First suppose a particle is moving in the above spacetime at nonrelativistic speeds (e.g. a planet orbiting the Sun) – specifically we assume $v = |dx^i/dt| \ll 1$. Show that to first order in $h_{\alpha\beta}$ and v, the equation of motion for a geodesic can be written as

$$\frac{d^2x^i}{dt^2} = -\frac{\partial\Phi}{\partial x^i} - \frac{\partial A_i}{\partial t}. \tag{3}$$

Thus Φ (the time-time part of the metric) acts like a gravitational potential: freely falling particles experience an apparent acceleration toward lower potential.

(b) Suppose that the spacetime is stationary, i.e. the metric can be written such that $g_{\alpha\beta}$ does not depend on t. Now consider an observer following the non-geodesic trajectory $x^i(t) = X^i$ (where X^i is a fixed set of coordinates). What is the relation between coordinate time t and the proper time τ measured by such an observer? If Alice and Bob are stationed at two different points in space, and both at rest in the above coordinate system, they may compare the rate of ticking of Alice's clock to Bob's clock, $d\tau_{\rm Alice}/d\tau_{\rm Bob}$; what is this? Can the "absolute" potential Φ be measured in this way?

(c) Now consider the metric given by

$$ds^2 = -(1 + ax^3)^2\, dt^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2, \tag{4}$$

where a is a positive constant. For $|x^3| \ll a^{-1}$, this may be treated as a small perturbation around Minkowski space; describe the implications of your results in (a) and (b) for this spacetime. What is the physical meaning of a?

(d) Show that Eq. (4) is in fact equivalent to Minkowski spacetime via a change of coordinates.[1] What is the trajectory of the observer stationed at $x^1(t) = x^2(t) = x^3(t) = 0$ in the Minkowski spacetime coordinates, and what is the magnitude of their 4-acceleration?

2. Orbits around a black hole. [18 points]

Consider the Schwarzschild metric below, which corresponds to the geometry outside a spherically symmetric object (planet, star, nonrotating black hole ...):

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - 2M/r} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{5}$$

We will only consider the $r > 2M$ region here (i.e. outside the event horizon).

[1] There are many possible answers – you only need to give one.

Warning: While we have labeled the radial coordinate "r," it does not correspond to a physically measurable radius – you can't stick a measuring tape down to the center of a black hole and count the number of centimeters. It is a coordinate chosen for computational convenience. We will encounter other radial coordinates later in the course with their own advantages.

(a) Compute the metric tensor components $g_{\alpha\beta}$ and the Christoffel symbols $\Gamma^{\alpha}_{\beta\gamma}$.

(b) We are interested in the subject of orbits around such an object. For the purposes of this problem, let us restrict ourselves to circular, equatorial orbits.[2] "Circular" here means that the particle trajectory is $r =$ constant; "equatorial" means that $\theta = \pi/2$. What conditions on the contravariant 4-velocity u^{α} do the circular + equatorial restrictions imply?

(c) Using the equations of geodesic deviation, show that circular equatorial orbits in the above spacetime imply that the coordinate components u^t and u^{ϕ} are constant. Moreover, if we require the orbits to be future-directed and prograde ($u^{\phi} > 0$), then at a given r there is at most one possible value of u^t and u^{ϕ}. Solve for the angular frequency $\Omega = d\phi/dt$ of the orbit. How does it compare to the familiar Keplerian result? And more importantly, to what measurement by a distant external observer does Ω actually correspond?

(d) The analysis from (c) should imply that there is a critical value of r within which circular orbits do not exist. What is this value? [**Note**: I'm not asking you to find the innermost stable circular orbit, beloved in high-energy astrophysics – I'm asking you to find the innermost circular orbit, without regard to stability.]

[2] The equatorial restriction actually doesn't do anything since the object is spherically symmetric, but it dramatically simplifies the algebra.