Ph 236 – Homework 3
Due: Wednesday, October 24, 2012

1. Relativistic shocks [18 points]
Relativistic shocks are believed to occur in many high-energy astrophysical phenomena (e.g. gamma ray bursts). Here you will work through the properties of a simplified model.

We consider a plane-parallel shock and work in its rest frame. The shock is located at $x = 0$ at all times, and is translation invariant in y and z. An ideal fluid with rest-frame baryon number density n, energy density ρ, pressure p, and 3-velocity β enters the shock from the $x < 0$ side. It emerges on the $x > 0$ side with baryon number density n', energy density ρ', etc.

You will want to define $\gamma = (1-\beta^2)^{-1/2}$ and $\gamma' = (1-\beta'^2)^{-1/2}$. Also you will use the rest mass per baryon of cold matter, $m_0 = 1.67 \times 10^{-24}$ g.

(a) Using the conservation of baryon number and energy-momentum, prove the following relations among the flow parameters:

$$\begin{aligned} n\beta\gamma &= n'\beta'\gamma' \\ (\rho + p)\beta\gamma^2 &= (\rho' + p')\beta'\gamma'^2 \\ (\rho\beta^2 + p)\gamma^2 &= (\rho'\beta'^2 + p')\gamma'^2. \end{aligned} \tag{1}$$

(b) A common situation is that the incoming gas is cold ($kT \ll \rho\beta^2$, so that initial thermal motions can be neglected) and the outgoing shocked gas is a radiation-dominated plasma – i.e. its stress-energy tensor is dominated by the sum of the baryonic rest mass and the radiation generated in the shock. Express ρ, p, and ρ' in terms of n, n', p', and m_0.

(c) Using your answer from part (b) and the first two equations from part (a), eliminate ρ, p, and ρ', and n', and express p' in terms of β'.

(d) Using the remaining equation from part (a), show that the post-shock velocity is related to the pre-shock velocity by the equation:

$$4\beta = -\frac{\sqrt{1-\beta'^2}}{\gamma\beta'} + 3\beta' + \frac{1}{\beta'}. \tag{2}$$

This equation always has two solutions, only one of which can represent a shock. Explain why.

(e) Plot the relation from part (d) using your favorite method.[1] Then find the behavior of β' in the two limiting cases:

- The *non-relativistic shock*, $\beta \to 0$. (I'm looking for β' to first order in β.) This describes a supernova shock as it propagates through the envelope of the dying star.
- The *ultra-relativistic shock*, $\beta \to 1$.

(f) Now transform to the rest frame of the upstream (unshocked) gas. In the ultrarelativistic case, what is the relation between the Lorentz factor of the shock and that of the postshock gas?

2. Electrodynamics based on the action. [18 points]
This problem works through the canonical formulation of electrodynamics. We begin with the action (in CGS units):

$$S = S_{\rm EM} + S_{\rm particle} + S_{\rm int}. \tag{3}$$

Here the purely electromagnetic part of the action is

$$S_{\rm EM} = -\frac{1}{16\pi}\int F_{\mu\nu}F^{\mu\nu}\, d^4x, \tag{4}$$

[1] Anything is fine here – gnuplot, Mathematica, slide rule + graph paper ...

where the integral is over all of spacetime; the particle part of the action for a particle of mass m is

$$S_{\text{particle}} = -m \int d\tau, \tag{5}$$

i.e. this is the mass of the particle times the length of its world line; and the interaction term is

$$S_{\text{int}} = \int A_\mu J^\mu \, d^4x. \tag{6}$$

As usual, the action is to be minimized with respect to $A_\mu(x^\alpha)$ and the trajectory of the particle, subject to the initial and final field configurations, and $\boldsymbol{F} \equiv \boldsymbol{dA}$.

(a) Prove that under gauge transformations, $A_\mu \to A_\mu + \chi_{,\mu}$ (where χ is an arbitrary scalar field), the change in the action ΔS can be written as a boundary integral only if $J^\mu{}_{,\mu} = 0$. Thus the conservation of charge is necessary in order for the action to be gauge invariant.

(b) Now suppose the particle has charge e. Show that the particle and the interaction terms can be written as

$$S_{\text{particle}} + S_{\text{int}} = \int \left(-m\sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}} + eA_\mu \frac{dx^\mu}{d\lambda} \right) d\lambda, \tag{7}$$

where λ is a parameter. By varying the functions $x^\mu(\lambda)$, show that

$$0 = -m\frac{du_\mu}{d\lambda} + eF_{\mu\nu}\frac{dx^\nu}{d\lambda}, \tag{8}$$

and hence derive the Lorentz force law.

(c) Show that varying the electromagnetic 4-potential $A_\mu(x^\alpha)$ leads to the Gauss-Ampère law $F^{\alpha\beta}{}_{,\beta} = 4\pi J^\alpha$.