Ph 236 – Homework 2

Due: Wednesday, October 17, 2012

1. Basis independence of contractions. [6 points]

Suppose that $\mathbf{S}$ is a rank $\binom{2}{1}$ tensor. Show that the contraction $\mathbf{T}$ of $\mathbf{S}$, defined as the rank $\binom{1}{0}$ tensor:

$$\mathbf{T}(\tilde{\boldsymbol{k}}) \equiv \sum_{\alpha=0}^{3} \mathbf{S}(\tilde{\boldsymbol{\omega}}^{\alpha}, \tilde{\boldsymbol{k}}, \boldsymbol{e}_{\alpha}) \tag{1}$$

is independent of the choice of basis.

2. Forms in 2 dimensions. [12 points]

Consider the 2-dimensional Euclidean space $\mathbb{R}^2$. Use the standard Euclidean metric so that vectors and 1-forms can be treated as interchangeable.

(a) Show that the dual of a scalar f is $\star f = f\boldsymbol{\epsilon}$. Describe geometrically what operation corresponds to taking the dual of a vector $\star\boldsymbol{v}$. Show that $\star\star f = f$ but $\star\star\boldsymbol{v} = -\boldsymbol{v}$.

(b) Now consider a vector field $\boldsymbol{v}(\boldsymbol{x})$. We may construct two scalar fields, $f = \star\boldsymbol{dv}$ and $h = \star\boldsymbol{d}(\star\boldsymbol{v})$. These should correspond to familiar operations in $\mathbb{R}^2$ – what are they?

3. Existence of 1-form potentials. [12 points]

A 2-form $\boldsymbol{F}$ is defined to be *closed* if $\boldsymbol{dF} = 0$. In this problem, you will consider the properties of closed 2-forms in n-dimensional spaces of simple topology, and show ultimately that they can be written as exterior derivatives of 1-forms: $\boldsymbol{F} = \boldsymbol{dA}$.

All of your proofs in this problem should be valid for any signature of the metric (we will use Latin indices for the purposes of the problem). However, you may assume a global coordinate system[1] $(x^1 ... x^n)$, and of course that $n \geq 2$.

The ultimate purpose of this exercise is as a prelude to Problem #4 on E&M, although the problems can be worked independently.

(a) Show that if there is a closed 2-form field $\boldsymbol{H}$ satisfying $H_{ni} = 0$ (i.e. last row vanishes) at all positions in $\mathbb{R}^n$, and $H_{ij} = 0$ on the plane $x^n = 0$ (i.e. on $\mathbb{R}^{n-1}$), that $\boldsymbol{H} = 0$ everywhere.

(b) Suppose that $n = 2$. Prove the existence of a 1-form $\boldsymbol{A}$ with $\boldsymbol{F} = \boldsymbol{dA}$.

(c) Now let us generalize (b) by means of mathematical induction. Suppose that we have a closed 2-form $\boldsymbol{F}$, and a 1-form field σ_i defined on the plane $x^n = 0$ (i.e. on $\mathbb{R}^{n-1}$) with $\sigma_{j,i} - \sigma_{i,j} = F_{ij}$ for $1 \leq i, j \leq n-1$. With the help of part (a), write down a 1-form $\boldsymbol{A}$ with $\boldsymbol{F} = \boldsymbol{dA}$.

(d) Is the 1-form $\boldsymbol{A}$ with $\boldsymbol{F} = \boldsymbol{dA}$ unique?

4. Electromagnetic potential. [6 points]

In class we showed that the electromagnetic field $\boldsymbol{F}$ is exactly a closed 2-form. Let us now split $\boldsymbol{A}$ (the 1-form potential – see Problem #3) into its time component $\Phi \equiv A_0$ and the spatial components A_i. Express the electric and magnetic field components E_i and B_i in terms of Φ and A_i. Show that you recover the usual expression for the field in terms of the electric potential Φ and magnetic potential A_i.

[1] If this assumption is relaxed, the existence of a globally defined vector potential $\boldsymbol{A}$ no longer holds. This has interesting implications for spacetimes of nontrivial topology, but we won't need to consider these yet.