Ph 236 – Homework 1
Due: Wednesday, October 10, 2012

1. Superluminal motion. [20 points]
One of the most spectacular observations from long-baseline radio interferometry is apparent *superluminal motion*. This occurs when the transverse motion of an object (as measured by $V_\perp = D\dot{\theta}$, where D is the distance to the object and $\dot{\theta}$ is the proper motion) exceeds the speed of light. Superluminal motion was first observed in the jets of the quasar 3C 273 in 1969, and has since been observed in a number of other sources.

This phenomenon seems impossible but is in fact entirely consistent with special relativity. You will explain it in this problem. For simplicity, you may ignore issues of the cosmological expansion etc., and treat the quasar and observer as being at rest relative to each other in a background Minkowski spacetime. The quasar-observer distance is much greater than the separation of any components in the quasar.

(a) Suppose that in the lab (unprimed) frame, the quasar is placed at the origin and the observer far away in the +1 coordinate direction. Show that if a blob at coordinates (x^0, x^1, x^2, x^3) emits light that is received later by the observer, that light is received at a time given by $x^0 - x^1$+constant.

Without loss of generality, we will orient our coordinate system so that the blob is ejected in the plane $x^3 = 0$, and in the direction with $x^2 > 0$. We will now define new coordinate system,

$$x^{0'} = x^0 - x^1, \quad x^{1'} = x^0 + x^1, \quad x^{2'} = x^2, \quad \text{and} \quad x^{3'} = x^3, \tag{1}$$

so that the time of arrival of light is $x^{0'}$ (plus an irrelevant constant). This new system is known as the *light-cone coordinate system*.

(b) Express the metric tensor $g_{\alpha'\beta'}$ in the primed coordinate system.

(c) Suppose that we parameterize the blob's trajectory in terms of $x^{0'}$. What is $V_\perp$ in terms of $dx^{1'}/dx^{0'}$ and $dx^{2'}/dx^{0'}$?

(d) What conditions on $dx^{1'}/dx^{0'}$ and $dx^{2'}/dx^{0'}$ must be satisfied in order for the blob's 4-velocity to be timelike? Show that this condition, combined with the formula in (c), allows solutions in which $V_\perp$ takes on any non-negative value.

(e) Now suppose that a value of $V_\perp$ is measured in some source. Write a formula for the 4-velocity components $u^{\alpha'}$ of the object in terms of $V_\perp$ and $q \equiv dx^{1'}/dx^{0'}$.

(f) Convert the 4-velocity back to the lab frame (i.e. obtain u^α). Show that the Lorentz factor γ of the blob in this frame in terms of q and $V_\perp$ is

$$\gamma = \frac{1+q}{2\sqrt{q - V_\perp^2}}. \tag{2}$$

Show that the minimum Lorentz factor for superluminal motion ($V_\perp \geq 1$) that satisfies all constraints is $\gamma = \sqrt{2}$, corresponding to a lab frame velocity of the blob of $1/\sqrt{2}$.

2. Normal forms of the metric tensor. [16 points]
This problem will justify the claims made in lecture (Lecture Notes I, §IIIB) about the possible forms of the symmetric tensor $g_{\mu\nu}$. Assume an n-dimensional space(time). In this problem any real basis transformation with invertible transformation matrix $\mathbf{L} \in \mathrm{GL}(n, \mathbb{R})$ is legal; complex transformations are disallowed.

(a) Prove that there exists a basis transformation that diagonalizes $g_{\mu\nu}$.

(b) Prove that a further basis transformation can set all diagonal entries equal to -1, 0, or $+1$ (while leaving $g_{\mu\nu}$ diagonal). In such a case, each basis vector can be categorized as either having negative, zero, or positive square-norm.

Let us now call the number of -1, 0, and $+1$ diagonal entries n_-, n_0, and n_+ (with $n_- + n_0 + n_+ = n$). We will now prove that the signature (n_-, n_0, n_+) is unique. Suppose that there were two different choices of basis (unprimed and primed) satisfying part (b), *but* with a different signature $(n_-, n_0, n_+) \neq (n'_-, n'_0, n'_+)$. Call the basis transformation matrix $\mathbf{L}$.

(c) Suppose that we had $n_- > n'_-$. Then prove that this implies that there is a nonzero vector $\boldsymbol{v}$ that satisfies the following conditions: [I] $v^{\alpha'} = 0$ for α' corresponding to any primed basis vector of negative square norm; and [II] $v^\beta = 0$ for β corresponding to any unprimed basis vector of zero or positive square norm. [*Hint*: consider the number of conditions imposed on $\boldsymbol{v}$ and the number of degrees of freedom.] Prove that calculations in the unprimed and primed bases lead to contradictory conclusions regarding the sign of $\boldsymbol{v} \cdot \boldsymbol{v}$, and hence that our assumption was impossible.

(d) Explain why arguments similar to those in (c) tell us that we must have $n'_- = n_-$, $n'_+ = n_+$. Then prove that $n'_0 = n_0$.