Ph 236 – Final Exam, Fall Term

Due: Wednesday, December 12, 2012

1. Falling into a black hole. [12 points]
Consider a Schwarzschild (non-rotating) black hole of mass M, which has metric:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - 2M/r} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2); \tag{1}$$

we will consider the region $r > 2M$. This system is manifestly spherically symmetric, time-independent, and asymptotically flat. The metric components $g_{\mu\nu}$ diverge as $r \to 2M$ (the event horizon). It is not obvious from the form of the metric whether this reflects a true singularity or merely a singularity of the coordinate system (next term we will learn it is a coordinate singularity).

(a) Consider the trajectory of a photon propagating radially outward ($p^r > 0$, $p^\theta = p^\phi = 0$). Use the null condition to find dt/dr as a function of r. Show that the amount of *coordinate time* required for the photon to travel from r_1 to r_2 is

$$\Delta t = r_2 - r_1 + 2M \ln \frac{r_2 - 2M}{r_1 - 2M}. \tag{2}$$

Show that this becomes infinite as $r_1 \to 2M$ – one manifestation of the black hole's event horizon.

(b) Now let us consider a daring observer $\mathcal{O}$ who decides to free-fall radially into the black hole ($u^r < 0$, $u^\theta = u^\phi = 0$). Explain why u_t is conserved on the observer's trajectory. Assign it the value $u_t = -\tilde{E}$; in one sentence, what is the physical interpretation of $\tilde{E}$?

(c) Find dt/dr and $d\tau/dr$ as functions of r and the constant E. Show that $\mathcal{O}$ reaches the event horizon in finite proper time τ, but in infinite coordinate time t. [You don't need to actually do the integral for τ, just prove it is finite.]

2. Energy in gravitational waves. [18 points]
This problem works through an alternate derivation of the fact that gravitational waves have an apparent energy density. The idea is that we will use the Raychaudhuri equation to show that – if one works beyond linear GR perturbation theory – neighboring particles are pulled together if a gravitational wave passes over them.

The setup for this problem will begin in a small region of flat spacetime, far removed from any sources, where we will build the usual set of Minkowski coordinates, and a congruence of geodesics with $u^\alpha = (1, 0, 0, 0)$, and consider in particular an observer who follows the geodesic that passes through the origin. We will then let a distant source (or sources) emit some gravitational waves that pass over the observer. The problem will be to determine what happens to the geodesics near the observer as a result. Assume that in some neighborhood of the observer the stress-energy tensor is zero.

(a) Show that for the above geodesics, the expansion $\theta = u^\alpha{}_{;\alpha}$ *exactly* satisfies

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma_{\alpha\beta}\sigma^{\alpha\beta}, \tag{3}$$

where $\sigma_{\alpha\beta}$ is the traceless-symmetric velocity gradient (as defined in Lecture XIV), and that the initial condition is $\theta(\tau_{\rm i}) = 0$ (where $\tau_{\rm i}$ is chosen to be before the gravitational waves arrive).

In what follows we will be interested in solving for θ to second order. To do so, we will need to solve for $\sigma_{\alpha\beta}$ to first order. We will do so in the transverse-traceless gauge,

$$ds^2 = -dt^2 + \left(\delta_{ij} + h^{\rm TT}_{ij}\right) dx^i\, dx^j + \mathcal{O}(h^2). \tag{4}$$

(b) Show that the curves x^i = constant, $t = \tau$ are geodesics. Thus, to first order in h, this is our congruence of geodesics.

(c) Find the shear to linear order in h.

(d) Using your result from parts (a) and (c), show that to linear order in h the expansion θ remains zero, and that to second order we have

$$\theta = -\frac{1}{4}\int_{\tau_i}^{\tau} \dot{h}_{ij}^{\mathrm{TT}}\dot{h}^{\mathrm{TT}\,ij}\,dt + \mathcal{O}(h^3). \tag{5}$$

Thus after the gravitational wave(s) pass by, the geodesics are converging – i.e. test particles initially at rest relative to the observer now have a net inward velocity with the apparent "$-\mathrm{div}\,\boldsymbol{v}$" given by the answer to part (d).

(e) Show that the velocity gradient in part (d) is equivalent to that one would find if instead of gravitational waves there was matter present with

$$\rho_{\mathrm{eff}} + 3p_{\mathrm{eff}} = \frac{1}{16\pi}\langle \dot{h}_{ij}^{\mathrm{TT}}\dot{h}^{\mathrm{TT}\,ij}\rangle, \tag{6}$$

where the average is taken over time. This is exactly what one would calculate using the "effective stress-energy of gravitational waves" described in class.