Lecture XXXVIII: Primordial perturbations: tensor sector

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I. INTRODUCTION

Our final step in the analysis is to consider the generation of perturbations in inflation.

The recommended reading if you like this subject is Liddle & Lyth, *Cosmological inflation and large scale structure*, Chapter 7.

II. SETUP NOTES

As a reminder, in inflation all patches start in causal contact and then exit the horizon. This is a consequence of the nature of the conformal time: $\int dt/a$ is finite in the future but (maybe) not the past. We ordinarily take the range $-\infty < \eta < 0$.

We will take $\hbar = 1$ in these notes.

III. THE HAMILTONIAN

Our first step is to build the Hamiltonian for perturbations in the scalar field. This is relatively straightforward: the Lagrangian is

$$L_\phi = \int \left[-\frac{1}{2} g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - V(\phi) \right] \alpha\, \gamma^{1/2}\, d^3x, \tag{1}$$

or in terms of our perturbation variables, setting $\phi(\eta, x^k) = \bar\phi(\eta) + \delta\phi(\eta, x^k)$:

$$\begin{aligned} L_\phi &= a^4 \int \left[\frac{1}{2} a^{-2}(1+A)^{-2}(\dot{\bar\phi} + \delta\dot\phi)^2 - a^{-2}\dot{\bar\phi} B_i \delta\phi_{,i} - \frac{1}{2} a^{-2}\delta\phi_{,i}\delta\phi_{,i} - V(\bar\phi) - V'(\bar\phi)\delta\phi - \frac{1}{2}V''(\bar\phi)\delta\phi^2 \right](1+A) \\ &\quad \times \left[1 - \frac{1}{2}h_{ii} + \frac{1}{8}(h_{ii}h_{jj} - 2h_{ij}h_{ij}) \right] d^3x. \end{aligned} \tag{2}$$

There is a conjugate momentum $\varpi(\eta, x^k) = \delta L_\phi / \delta(\delta\dot\phi(\eta, x^k))$:

$$\varpi = a^2(1+A)^{-1}(\dot{\bar\phi} + \delta\dot\phi)\left[1 - \frac{1}{2}h_{ii} + \frac{1}{8}(h_{ii}h_{jj} - 2h_{ij}h_{ij}) \right]; \tag{3}$$

then the Hamiltonian for the scalar field is

$$\begin{aligned} H_\phi &= a^4 \int \left[\frac{1}{2} a^{-6}\varpi^2 + a^{-2}\dot{\bar\phi} B_i \delta\phi_{,i} + \frac{1}{2} a^{-2}\delta\phi_{,i}\delta\phi_{,i} + V(\bar\phi) + V'(\bar\phi)\delta\phi + \frac{1}{2}V''(\bar\phi)\delta\phi^2 \right](1+A) \\ &\quad \times \left[1 - \frac{1}{2}h_{ii} + \frac{1}{8}(h_{ii}h_{jj} - 2h_{ij}h_{ij}) \right] d^3x - \dot{\bar\phi} \int \varpi\, d^3x. \end{aligned} \tag{4}$$

Now ϖ has a mean value $\bar\varpi = a^2\dot{\bar\phi}$. We may take this out by the canonical transformation to $\delta\varpi = \varpi - \bar\varpi$, and with a corresponding change to the Hamiltonian that involves only the zero mode. Then

$$H_\phi = a^4 \int \left[\frac{1}{2} a^{-2}\dot{\bar\phi}^2 + a^{-4}\dot{\bar\phi}\delta\varpi + \frac{1}{2}a^{-6}\delta\varpi^2 - a^{-2}\dot{\bar\phi} B_i\delta\phi_{,i} + \frac{1}{2}\delta\phi_{,i}\delta\phi_{,i} - V(\bar\phi) - V'(\bar\phi)\delta\phi - \frac{1}{2}V''(\bar\phi)\delta\phi^2 \right](1+A)$$

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$$\times\left[1-\frac{1}{2}h_{ii}+\frac{1}{8}(h_{ii}h_{jj}-2h_{ij}h_{ij})\right]d^3x-\dot{\bar\phi}\int\delta\varpi\,d^3x\;+\text{zero mode.}\tag{5}$$

Equation (5) is exact, at least in terms of linear perturbation theory. In slow roll inflation, however, the terms involving the derivative of the potential – and consequently those involving $\dot{\bar\phi}$ – are small. Then one may write

$$\begin{aligned}H_\phi\approx H_\phi^{\rm slow} &= \int\left[\frac{\delta\varpi^2}{2a^2}+\frac{1}{2}a^4\delta\phi_{,i}\delta\phi_{,i}\right]d^3x\\ &\quad -V(\bar\phi)a^4\int(1+A)\left[1-\frac{1}{2}h_{ii}+\frac{1}{8}(h_{ii}h_{jj}-2h_{ij}h_{ij})\right]d^3x.\end{aligned}\tag{6}$$

In Fourier space, we have the canonically conjugate pair

$$\delta\phi(x)=\int\delta\phi(k)e^{ikx}\,\frac{d^3k}{(2\pi)^3}\quad\leftrightarrow\quad\delta\varpi(x)=\int\delta\varpi(k)e^{-ikx}\,d^3k.\tag{7}$$

Then:

$$\begin{aligned}H_\phi^{\rm slow} &= \int\left[\frac{(2\pi)^3}{2a^2}\delta\varpi(k)\delta\varpi(-k)+\frac{a^4k^2}{2(2\pi)^3}\delta\phi(k)\delta\phi(-k)\right]d^3k\\ &\quad -V(\bar\phi)a^4\int(1+A)\left[1-\frac{1}{2}h_{ii}+\frac{1}{8}(h_{ii}h_{jj}-2h_{ij}h_{ij})\right]d^3x.\end{aligned}\tag{8}$$

Equation (8) will be the basis for our consideration of the perturbations generated during inflation. We consider both the scalar (next lecture!) and tensor sector. Note that in scalar field inflation there are no vector perturbations at all, since we have only one canonically conjugate pair $(h_{\rm H},\kappa_{\rm H})$ for each polarization (horizontal or vertical), the constraint specifies $\kappa_{\rm H}$ in terms of $h_{\rm H}$, and a gauge transformation can set $h_{\rm H}=0$.

IV. TENSOR PERTURBATIONS

We consider the tensor perturbations first, as these are the simplest to understand. The Hamiltonian for the tensors is equivalent to that in the ΛCDM sector, except that $V(\bar\phi)$ replaces Λ.

It will be seen that the development of tensor perturbations depends only on the dynamics near horizon exit. We therefore further replace $V(\bar\phi)$ with the value of the potential at the reference field ϕ_e, corresponding to the field value at horizon exit – i.e. when $\mathcal{H}=k$. (Remember that during inflation, $\mathcal{H}\approx-1/\eta$ is exponentially increasing with time.)

The Hamiltonian can be obtained from Lecture XXXVI, Eq. (44), and with the replacements

$$\mathcal{H}=-\frac{1}{\eta}\quad\text{and}\quad\frac{5a^2\mathcal{H}^2-a^4\Lambda}{16\pi}\to\frac{5a^2\mathcal{H}^2-a^4(3a^{-2}\mathcal{H}^2)}{16\pi}=\frac{a^2\mathcal{H}^2}{8\pi}=\frac{a^2}{8\pi\eta^2}.\tag{9}$$

Then:

$$\begin{aligned}H^{\rm tensor} &= \int\Big[\frac{a^2k^2}{32\pi(2\pi)^3}h_+(k)h_+(-k)+\frac{32\pi(2\pi)^3}{a^2}\kappa_+(k)\kappa_+(-k)+\frac{8}{\eta}h_+(k)\kappa_+(k)\\ &\quad +\frac{a^2}{8\pi(2\pi)^3\eta^2}h_+(k)h_+(-k)\Big]d^3k+[+\leftrightarrow\times].\end{aligned}\tag{10}$$

A. Classical analysis

We now consider the classical evolution under the tensor Hamiltonian.

There is a Poisson bracket:

$$\{h_+(k),\kappa_+(k')\}_{\rm P}=\frac{1}{2}\delta^{(3)}(k-k').\tag{11}$$

There is also a classical solution: Lecture XXXVI gives the ODE (with $\mathcal{H}=-1/\eta$)

$$\ddot h_+(k)-2\eta^{-1}\dot h_+(k)+k^2h_+(k)=0.\tag{12}$$

The solution is standard: we note that if $h_+(k) = \eta^2 u$, then

$$\eta^2 \ddot{u} + 2\eta\dot{u} + 2u + (k^2\eta^2 - 2)u = 0, \tag{13}$$

which has the two linearly independent solutions $u = j_1(k\eta)$ and $y_1(k\eta)$ (spherical Bessel functions). Thus

$$h_+(k,\eta) = Z_1(k)(\sin k\eta - k\eta\cos k\eta) + Z_2(k)(-\cos k\eta - k\eta\sin k\eta). \tag{14}$$

Clearly the solution oscillates at early times with amplitudes determined by $Z_1(k)$ and $Z_2(k)$, and asymptotes at late times to $-Z_2(k)$ (the Z_1 solution is decaying).

We also see in this case that the conjugate momentum satisfies

$$\kappa_+(k,\eta) = \frac{(2\pi)^3 a^2}{32\pi}[\dot{h}_+(-k) + 4\mathcal{H}h_+(-k)], \tag{15}$$

and substituting in the solution from Eq. (14), combined with $\mathcal{H} = -1/\eta$ and

$$a^2 = \frac{\mathcal{H}^2}{H^2} = \frac{1}{H_{\rm I}^2\eta^2} \tag{16}$$

(where $H_{\rm I}$ is the Hubble rate during inflation) gives

$$\kappa_+(k,\eta) = \frac{1}{32\pi(2\pi)^3 H_{\rm I}^2\eta^3}\left[Z_1(-k)[(k^2\eta^2-4)\sin k\eta + 4k\eta\cos k\eta] + Z_2(-k)[-(k^2\eta^2-4)\cos k\eta + 4k\eta\sin k\eta]\right]. \tag{17}$$

B. Quantum analysis

Up to now, everything has been classical. In classical physics, it would have been perfectly fine to say that there were no initial gravitational waves: then $Z_1(k) = Z_2(k) = 0\,\forall k$. Quantum mechanically, however, there is a problem: one cannot set both a coordinate $h_+(k)$ and its conjugate momentum $\kappa_+(k)$ to zero.

To see how to proceed in this situation, we re-interpret $Z_1(k)$ and $Z_2(k)$ as quantum operators, and take $h_+(k,\eta)$ and $\kappa_+(k,\eta)$ to be operators in the Heisenberg picture of quantum mechanics (operators evolve, wave functions don't). Then there should be some commutation relation

$$[Z_a(k), Z_b(k')] = S_{ab}(k)\delta^{(3)}(k+k') \tag{18}$$

(the δ-function is by translational invariance) and $a, b = 1, 2$. Symmetry considerations force S_{ab} to depend only on the magnitude of k and to be antisymmetric – thus $S_{11}(k) = S_{22}(k) = 0$ and $S_{12}(k) = -S_{21}(k)$. Inspection of Eq. (14) and Eq. (17) then shows that

$$[h_+(k,\eta), \kappa_+(k',\eta)] = \frac{1}{32\pi(2\pi)^3 H_{\rm I}^2\eta^3} S_{12}(k)\, k^3\eta^3\delta^{(3)}(k-k') = \frac{k^3}{32\pi(2\pi)^3 H_{\rm I}^2} S_{12}(k)\delta^{(3)}(k-k'). \tag{19}$$

Considering the Poisson bracket, Eq. (11), we want this to equal $\frac{1}{2}i\delta^{(3)}(k-k')$. Therefore, we must have

$$S_{12}(k) = i\frac{16\pi(2\pi)^3 H_{\rm I}^2}{k^3}. \tag{20}$$

That is, the commutation relation for the amplitudes is

$$[Z_a(k), Z_b(k')] = i\epsilon_{ab}\frac{16\pi(2\pi)^3 H_{\rm I}^2}{k^3}\delta^{(3)}(k+k'), \tag{21}$$

where ϵ_{ab} is the antisymmetric symbol.

The classical real nature of the gravitational wave amplitude means that the quantum operator is Hermitian; in Fourier space this means

$$Z_a^\dagger(k) = Z_a(-k). \tag{22}$$

We want to know what is the expectation value of the final gravitational wave amplitude. As $\eta \to 0$ (end of inflation), we have

$$h_+(k,0) = -Z_2(k) \tag{23}$$

and so

$$\langle h_+(k,0)h_+(k',0)\rangle = \langle Z_2(k)Z_2(k')\rangle. \tag{24}$$

But at this point we have a problem: we can't actually evaluate Eq. (24), because despite having a complete description of the operator, we don't have a description of the quantum state of the Universe! You might imagine that the Universe started out with "no gravitational waves," which seems like a reasonable first guess (and must be true if primordial gravitational waves are **only** those generated by inflation) – but what does that even mean?

C. The vacuum

A clue is that the evolution of each Fourier mode of the gravitational wave distribution is a quantum linear oscillator with time-dependent Hamiltonian. Normally in quantum mechanics of a particle, with a Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2, \tag{25}$$

we write the solutions as $x = C_1 f_1(t) + C_2 f_2(t)$, where f_1 and f_2 are functions of time:

$$x = C_1\frac{\cos\omega t}{(m\omega)^{1/2}} + C_2\frac{\sin\omega t}{(m\omega)^{1/2}}. \tag{26}$$

Then there is a corresponding function for the momentum:

$$p = -C_1(m\omega)^{1/2}\sin\omega t + C_2(m\omega)^{1/2}\cos\omega t. \tag{27}$$

Here we have normalized these solutions so that $[C_1, C_2] = i$. Reality conditions imply that C_1 and C_2 are Hermitian. Then we take the step of choosing an *annihilation operator* b and a *creation operator* $b^\dagger$ (the Hermitian conjugate) via

$$b = \frac{C_1 + iC_2}{\sqrt{2}}, \quad \text{and} \quad b^\dagger = \frac{C_1 - iC_2}{\sqrt{2}}. \tag{28}$$

These obey the relation $[b, b^\dagger] = 1$. We have

$$x = \frac{1}{(2m\omega)^{1/2}}(be^{-i\omega t} + b^\dagger e^{i\omega t}). \tag{29}$$

The Hamiltonian can be written as

$$H = \omega\left(b^\dagger b + \frac{1}{2}\right). \tag{30}$$

The ground state of the quantum harmonic oscillator is thus the vacuum $|\text{vac}\rangle$ defined by $b|\text{vac}\rangle = 0$. This really defines a state up to an irrelevant phase: this is because with the transition $p \to -i\partial_x$, the annihilation operator is a first-order differential operator and specifying a single amplitude $\psi(x=0)$ is then sufficient to solve the entire wave function. It can then be appropriately normalized.

How do we generalize this to the case of a quadratic, time-dependent Hamiltonian? Clearly, we have to make some choice of annihilation and creation operators – essentially, we must separate the solution for x into "positive frequency" and "negative frequency" modes – without the benefit of the clean complex-exponential time dependences (Eq. 29) that we get for constant Hamiltonian (and thus uniquely defined energy eigenstates for all time). The only non-negotiable aspect of this is the commutation relation $[b, b^\dagger] = 1$: there are thus many possible "vacuum" states corresponding to different choices of the operator b.

[Aside: This ambiguity does not exist in special relativity because a mode that is positive frequency complex exponential for one inertial observer is so for all. Therefore, the "vacuum" exists in special relativistic quantum field theory as a Lorentz-invariant quantum state. But this approach is not meaningful in curved spacetime.]

There is no general solution to this problem, but fortunately there is a special choice of vacuum – the *Bunch-Davies vacuum* – in inflating spacetimes. An investigation of Eq. (14) shows that at $-k\eta \gg 1$,

$$h_+(k,\eta) \sim -\frac{1}{2}k\eta[Z_1(k) + iZ_2(k)]e^{-ik\eta} - \frac{1}{2}k\eta[Z_1(k) - iZ_2(k)]e^{ik\eta}. \tag{31}$$

Thus we expect that $Z_1(k) + iZ_2(k)$ will play the role of an annihilation operator at early times, when the Fourier mode is inside the horizon and the gravitational wave propagates through many wavelengths before the background spacetime changes. In this sense, the system asymptotically looks like a gravitational wave propagating in Minkowski space at early times, and we treat the conventional flat-spacetime ground state as the vacuum. The specific annihilation operator that we need for proper normalization is

$$b(k) = \sqrt{\frac{k^3}{32\pi(2\pi)^3 H_{\rm I}^2}}[Z_1(k) + iZ_2(k)], \tag{32}$$

with

$$b^\dagger(k) = \sqrt{\frac{k^3}{32\pi(2\pi)^3 H_{\rm I}^2}}[Z_1(-k) - iZ_2(-k)]. \tag{33}$$

Then $[b(k), b^\dagger(k')] = \delta^{(3)}(k - k')$. The Bunch-Davies vacuum is the state defined by the annihilation operator giving zero:

$$b(k)|{\rm vac}\rangle = 0. \tag{34}$$

D. Primordial gravitational wave amplitude

We are finally ready to evaluate the primordial gravitational wave amplitude in the Bunch-Davies vacuum. This begins by using Eq. (32) to get

$$Z_2(k) = \sqrt{\frac{8\pi(2\pi)^3 H_{\rm I}^2}{k^3}}[-ib(k) + ib^\dagger(-k)]. \tag{35}$$

Then from Eq. (24):

$$\langle h_+(k,0)h_+(k',0)\rangle = \langle Z_2(k)Z_2(k')\rangle = \frac{8\pi H_{\rm I}^2}{(2\pi)^3 k^{3/2}k'^{3/2}}\langle {\rm vac}|[-ib(k) + ib^\dagger(-k)][-ib(k') + ib^\dagger(-k')]|{\rm vac}\rangle. \tag{36}$$

By operating on the vacuum state with the annihilation operator, and back-acting on the bra state with the creation operator, we find that of the four terms in the expectation value all but one vanishes:

$$\langle h_+(k,0)h_+(k',0)\rangle = \frac{8\pi(2\pi)^3 H_{\rm I}^2}{k^{3/2}k'^{3/2}}\langle {\rm vac}|b(k)b^\dagger(-k')|{\rm vac}\rangle. \tag{37}$$

Next we use the commutation relation:

$$b(k)b^\dagger(-k') = b^\dagger(-k')b(k) + [b(k), b^\dagger(-k')] = b^\dagger(-k')b(k) + \delta^{(3)}(k + k'). \tag{38}$$

Using the annihilation property on the vacuum, we are left with

$$\langle h_+(k,0)h_+(k',0)\rangle = \frac{8\pi H_{\rm I}^2}{(2\pi)^3 k^3}\delta^{(3)}(k + k'). \tag{39}$$

It is convenient to phrase this in terms of the RMS strain of gravitational waves. By summing the two polarizations, we find that

$$\langle h_{ij}(k,0)h_{ij}(k',0)\rangle_{\rm tensor} = \frac{32\pi(2\pi)^3 H_{\rm I}^2}{k^3}\delta^{(3)}(k + k'), \tag{40}$$

where the subscript "tensor" reminds us that only the tensor (and not the scalar) contribution is included. Then using

$$h_{ij}(x,0) = \int h_{ij}(k,0)e^{ikx}\,\frac{d^3k}{(2\pi)^3}, \tag{41}$$

we find that the variance of the strain is

$$\langle h_{ij}h_{ij}(x,0)\rangle_{\rm tensor} = \int \langle h_{ij}(k,0)h_{ij}(k',0)\rangle_{\rm tensor}\frac{d^3k\,d^3k'}{(2\pi)^6} = \int \frac{32\pi H_{\rm I}^2}{(2\pi)^3k^3}\,d^3k. \tag{42}$$

Then the contribution per logarithmic range in k to the variance $\langle h_{ij}h_{ij}\rangle$ is $4\pi k^3$ times the integrand:

$$\frac{d\langle h_{ij}h_{ij}(x,0)\rangle_{\rm tensor}}{d\ln k} = \frac{128\pi^2 H_{\rm I}^2}{(2\pi)^3} = \frac{16H_{\rm I}^2}{\pi}. \tag{43}$$

There is therefore a strain of $h_{\rm rms}^2 = \frac{1}{8}\langle h_{ij}h_{ij}\rangle_{\rm tensor}$ or

$$\frac{d(h_{\rm rms}^2)}{d\ln k} = \frac{2H_{\rm I}^2}{\pi} = \frac{3}{4\pi^2}V(\phi_e). \tag{44}$$

This is the standard result for gravitational waves generated during inflation.

Note that the perturbation spectrum is logarithmically divergent: gravitational waves are generated on all scales. This is not a problem since the waves are "generated" (transition from adiabatic fluctuations to frozen in) at the time of horizon exit $\eta \sim -k^{-1}$, so that at short wavelengths there is a cutoff (but we may never find it). At wavelengths longer than the present-day horizon scale we cannot measure the gravitational waves.

The most promising way to detect inflationary GWs is through the polarization of the CMB. Here $\dot{h}_{ij}$ generates a quadrupole moment of the radiation field, which re-scatters off free electrons during the recombination epoch (or later, during reionization) to produce polarized radiation. The method is sensitive to wavenumbers of $k \sim 10^{-3.5} - 10^{-2}$ Mpc^{-1}.

The inflationary gravitational wave background, if detected, would provide a direct handle on $V(\phi_e)$ and hence the energy scale of inflation. This would be great, although it is not guaranteed: $V(\phi_e)$ could be so small as to be undetectable. If we were so lucky as to find inflationary gravitational waves, however, one might imagine taking the next step. Remember that ϕ_e varies slowly during inflation, so if we go one e-fold of expansion later (N decreases by 1) then the gravitational wave amplitude changes. Over a narrow range of wavenumbers one expects variation of the form

$$\frac{d(h_{\rm rms}^2)}{d\ln k} \propto k^{n_t} \tag{45}$$

for some exponent n_t (the *tensor spectral index*) where

$$n_t = \frac{d\ln V(\phi_e)}{d\ln k} = \frac{d\ln V(\phi_e)}{H_{\rm I}\,dt} = V'(\phi_e)\phi_{,t}H_{\rm I}V(\phi_e) = -[V'(\phi_e)]^2 3H_{\rm I}^2 V(\phi_e) = -[V'(\phi_e)]^2 8\pi[V(\phi_e)]^2 = -2\epsilon_{\rm V}. \tag{46}$$

Therefore the tensor spectral index, if ever measured, would provide information on not just the potential during inflation but its derivative (the slow-roll parameter).