Lecture XXXVII: Inflation

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I. INTRODUCTION

We are finally ready to consider the problem of cosmic *inflation* – a hypothesized period of accelerated expansion in the early Universe that would solve the horizon problem and provide a source for the observed perturbations.

The recommended reading if you like this subject is Liddle & Lyth, *Cosmological inflation and large scale structure*, Chapter 3.

II. REQUIREMENTS AND MOTIVATION FOR SCALAR FIELD INFLATION

The key requirement for inflation is to have a phase in the early Universe, prior to the radiation-dominated epoch, where distant regions, separated in comoving distance by of order the modern-day Hubble scale, would have been in causal contact. Therefore the conformal time during inflation would have to have achieved very large negative values. That is, the integral

$$\eta = \int \frac{dt}{a} \tag{1}$$

would have to have been unbounded at early t. If we recall that in a flat model $a \propto t^{2/[3(1+w)]}$, where $w = p/\rho$ is the equation of state, we see that

$$\eta \propto t^{1-2/[3(1+w)]} = t^{(1+3w)/[3(1+w)]} \propto a^{(1+3w)/2}. \tag{2}$$

Since η is an increasing function of a for an expanding universe, we have a negative coefficient allowing $\eta \rightarrow -\infty$, if $w < -1/3$. If the universe was dominated by stuff with this property, then it should have inflated.

Most of the types of matter fields we have considered thus far – certainly all the normal matter – has $w > -1/3$ and is not suitable for inflation. The cosmological constant ($w = -1$) is suitable for inflation: if it dominates, we have $H = H_\Lambda = (\Lambda/3)^{1/2}$ and hence

$$a = a_1 e^{H_\Lambda t} \quad \rightarrow \quad \eta = -\frac{1}{a_1 H_\Lambda} e^{-H_\Lambda t}. \tag{3}$$

Unfortunately the present-day cosmological constant (with $H_\Lambda^{-1} \sim 10^{10}$ years) would have been negligible in the early Universe, so it doesn't solve the horizon problem. Rather, some other stuff that behave similar to Λ must have existed, and then disappeared. There are a large number of ways this might have happened, but clearly at least one dynamical degree of freedom is required. The simplest possibility is to invoke a scalar field ϕ, with action:

$$S_\phi = \int \left[-\frac{1}{2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - V(\phi) \right] \sqrt{-g}\, d^4x. \tag{4}$$

(Recall that for a scalar the partial and covariant derivative are equivalent.) The stress-energy tensor for the scalar field is

$$\begin{aligned} T^{\mu\nu} = 2(-g)^{-1/2} \frac{\delta S_\phi}{\delta g_{\mu\nu}} &= 2(-g)^{-1/2} \left[\frac{1}{2} g^{\mu\alpha} g^{\nu\beta} \phi_{,\alpha} \phi_{,\beta} (-g)^{1/2} + \frac{1}{2} [-\frac{1}{2} g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} - V(\phi)] g^{\mu\nu} (-g)^{1/2} \right] \\ &= g^{\mu\alpha} g^{\nu\beta} \phi_{,\alpha} \phi_{,\beta} - \frac{1}{2} g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} g^{\mu\nu} - V(\phi) g^{\mu\nu}. \end{aligned} \tag{5}$$

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For a spatially homogeneous scalar field in an FRW universe, the spatial derivatives vanish and we find

$$\rho = \frac{1}{2}(\phi_{,t})^2 + V(\phi) \quad \text{and} \quad p = \frac{1}{2}(\phi_{,t})^2 - V(\phi). \tag{6}$$

(Note that we inserted the $-$ sign in the kinetic energy term in S_ϕ in order to make the kinetic term in ρ positive.) If the potential term $V(\phi)$ dominates, then the scalar field will act like a cosmological constant. (The kinetic term does not do what we want: if it dominates then $w \approx 1$.) Unlike the true Λ, however, the potential will turn off if ϕ reaches the minimum of its potential and $V_{\rm min} \approx 0$.

Obviously, this leaves a few burning questions:

- Under what conditions does the potential energy term dominate?
- Why should the minimum of the potential be $V_{\rm min} \approx 0$, rather than some other value?
- What relation does the *inflation field* ϕ have to the Standard Model particles?
- How does the scalar field dominated model transition into the radiation-dominated phase that we have measured so accurately (BBN, CMB, etc.)?
- Aside from solving the horizon problem, what observable relics should be left today from the inflationary epoch?

Of these, the first is the easiest to answer, and we will do so shortly.

The issue of the minimum of the potential is a bit more subtle, since it is a matter of convention: clearly the action doesn't change if we set

$$V(\phi) \to V(\phi) + \Delta V \quad \text{and} \quad \Lambda \to \Lambda - \Delta V. \tag{7}$$

In other words, we may take $V_{\rm min} = 0$ and absorb all the constants of the inflation potential (and of the potential of any other fields that may exist) into Λ. Now of course this doesn't solve the real problem: we still have to explain why the sum of all these contributions is close to zero. If inflation occurred at an energy scale large compared to that in BBN, we would expect that $V(\phi) > (10\,\text{MeV})^4 \sim 10^{28}\,\text{eV}^4$ back then whereas the observed $\Lambda \sim 10^{-8}\,\text{eV}^4$, so having the true $V_{\rm min} + \Lambda$ as small as it is today suggests that there were > 32 decimal places of cancellation ... this usually doesn't happen by accident!

In fact, the situation here is far worse: in quantum field theory, Λ should also contain "loop corrections" from the zero-point energy of all of the fields present (photons, electrons ...). This should be of the order of

$$\Delta\Lambda \sim \int \frac{1}{2}\hbar\omega \frac{d^3p}{(2\pi)^3} \sim p_{\rm max}^4, \tag{8}$$

where $p_{\rm max}$ is the maximum momentum. If that's the Planck scale (a good guess) then $\Delta\Lambda \sim (10^{28}\,\text{eV})^4 \sim 10^{112}\,\text{eV}^4$. So Λ contains a correction that is 120 orders of magnitude larger than its observed value: this is known as the *cosmological constant problem*. It's fair to say that despite decades of trying, and plenty of innovative ideas, this problem remains unsolved. But the same problem of course exists without the inflaton. So at least inflation has not created any new problems; whether it has anything at all to do with the solution is an open issue.

The issue of how inflation relates to the Standard Model is not really the subject of this class (GR). Clearly a scalar ϕ can't be any of the known SM particles, none of which are scalar. The force carriers are vectors and could take on mean values in the universe – e.g. for the photons $\langle A_t \rangle \neq 0$ – but these can be gauged away. You might also imagine scalars made of pairs of fermions (analogous to Cooper pairs in a superconductor); it's not crazy to suggest inflation models based on something like this, but thermalized high-temperature plasma made of the known particles doesn't exhibit this behavior. Then there's the SM Higgs, which exhibits the correct behavior all the way up to the (proposed) form of its potential:

$$V(\phi) = V_0 - \frac{1}{2}\mu^2\phi^2 + \frac{1}{4}\lambda\phi^4, \tag{9}$$

but this technically doesn't work out: the perturbations it generates, for example, are way too small. So the inflaton has to be something new. A Higgs-like particle but at the GUT scale (breaking the symmetry between strong and electroweak interactions) might work, but is just one of many possibilities.

Regardless of what the inflaton was, and the energy scale $\rho^{1/4}$ at which inflation happened, there must be some interaction with the usual particles in order to match on to the observed features of the radiation-dominated epoch. The production of the CMB merely requires the inflaton to decay into SM particles (or other, related more massive particles) and have enough interactions to completely thermalize – a process known as *reheating*. The good news is that this is a simple and common outcome of particle decay. The bad news is that thermalization, by driving a system to the state of maximum entropy, erases all knowledge of how it got there, which is bad for those of us trying to probe the early Universe.

Another bit of bad news is that this argument says nothing about the production of net baryon number in the Universe (why are there more protons than antiprotons?) – one has to suppose that at sufficiently high temperatures, baryon number is not actually conserved. This doesn't worry some people; after all, most attempts to unify the fundamental forces predict some amount of baryon non-conservation (but the fact remains that this has never been observed). Moreover I haven't said anything yet about the production of dark matter, although depending on what the dark matter particle is, it might actually be a relic from the time when the Universe was hot and dense enough for it to interact nongravitationally. The neutrino background is an example of such a relic, and many viable dark matter candidates have been proposed along these lines, but we won't study this issue in this class.

At this point, with basically an entire new sector of physics invented to make inflation work, the last "burning issue" comes to the fore: what are the quantitatively testable predictions of inflation? The answer lies in the spectrum of perturbations it generates. But first we must consider the mean evolution.

III. HOMOGENEOUS EVOLUTION

We may use the continuity equation $\rho_{,t} = -3H(\rho + p)$ to derive an equation of motion for the scalar. This is

$$\partial_t \left[\frac{1}{2}(\phi_{,t})^2 + V(\phi) \right] = -3H(\phi_{,t})^2 \tag{10}$$

or

$$\phi_{,t}\phi_{,tt} + V'(\phi)\phi_{,t} = -3H(\phi_{,t})^2. \tag{11}$$

Solving for $\phi_{,tt}$, we find

$$\phi_{,tt} = -3H\phi_{,t} - V'(\phi). \tag{12}$$

The other equation we need is the Friedmann equation,

$$H = \sqrt{\frac{8\pi}{3}\left[\frac{1}{2}(\phi_{,t})^2 + V(\phi)\right]}. \tag{13}$$

Thus **the scalar field obeys a nonlinear second-order ODE**.

A closer inspection of Eq. (12) reveals that it is the equation for a particle moving in a potential $V(\phi)$ but with a strange "frictional" term $-3H\phi_{,t}$. This is known as *Hubble friction* and it is not real friction: like the adiabatic damping of gravitational waves, it would be reversed in a contracting universe. Nevertheless, the analogy immediately suggests that one might look for two limiting kinds of solutions: (i) one where the frictional term is subdominant, and the field value $\phi(t)$ oscillates like a particle in a conservative potential with slow damping as the Universe expands – i.e. $\phi_{tt} \approx -V'(\phi)$; and (ii) one where the frictional term is dominant, we have $3H\phi_{,t} \approx V'(\phi)$, and the equations of motion are effectively first order.

The parameter controlling the difference between these is the ratio of the Hubble time to the field evolution time. There are a few ways to quantify this, but a convenient one is

$$t_{\rm evol} = -\frac{\rho}{\rho_{,t}}. \tag{14}$$

Then we define

$$\epsilon_{\rm H} \equiv \frac{H^{-1}}{t_{\rm evol}} = -\frac{H^{-1}\rho_{,t}}{\rho} = 3\frac{(\rho+p)}{\rho} = 3\frac{(\phi_{,t})^2/2}{(\phi_{,t})^2/2 + V(\phi)}. \tag{15}$$

This is just 3 times the ratio of the kinetic to the total energy. If $\epsilon_{\rm H} \ll 1$, then we expect case (ii) to hold. Actually there is another condition that the motion of the field $\phi_{,t}$ be able to reach steady-state in less than a Hubble time, i.e.

$$\eta_{\rm H} \equiv -\frac{\phi_{,tt}}{H\phi_{,t}} \tag{16}$$

should also have $|\eta_{\rm H}| \ll 1$. If these conditions hold and we are in regime (ii) then we say that we are in *slow-roll inflation*. Note that $\epsilon_{\rm H} \ll 1$ automatically implies an inflating universe ($w < -1/3$).

IV. SLOW ROLL INFLATION

Our next step is to consider for what potentials slow-roll inflation can occur. The steady-state condition $|\eta_{\rm H}| \ll 1$ and the dominance oft he potential energy imply

$$\phi_{,t} \approx -\frac{V'(\phi)}{3H} \approx -\frac{V'(\phi)}{\sqrt{24\pi V(\phi)}}. \tag{17}$$

This is the slow-roll equation of motion and we will use it often – note that it is first order!

Under what circumstances is this picture consistent? Clearly we should be able to compute $\epsilon_{\rm H}$ and $\eta_{\rm H}$ in this approximation. We find

$$\epsilon_{\rm H} \approx 3\frac{(\phi_{,t})^2}{2V(\phi)} \approx \frac{[V'(\phi)]^2}{16\pi[V(\phi)]^2} \tag{18}$$

and

$$\eta_{\rm H} = -H^{-1}\partial_t \ln\phi_{,t} \approx \frac{V'(\phi)}{3H^2}\partial_\phi \ln\phi_{,t} \approx \frac{V'(\phi)}{8\pi V(\phi)}\partial_\phi\left[\ln V'(\phi) - \frac{1}{2}\ln V(\phi)\right] = \frac{V''(\phi)}{8\pi V(\phi)} - \frac{[V'(\phi)]^2}{16\pi[V(\phi)]^2}. \tag{19}$$

It is convenient to define the following parameters that depend only on the field value and not the dynamics:

$$\epsilon_{\rm V} \equiv \frac{[V'(\phi)]^2}{16\pi[V(\phi)]^2} \quad \text{and} \quad \eta_{\rm V} \equiv \frac{V''(\phi)}{8\pi V(\phi)}, \tag{20}$$

so that in slow roll $\epsilon_{\rm H} \approx \epsilon_{\rm V}$ and $\eta_{\rm H} \approx \eta_{\rm V} - \epsilon_{\rm V}$. The parameters $\{\epsilon_{\rm V}, \eta_{\rm V}, \epsilon_{\rm H}, \eta_{\rm H}\}$ are collectively known as *slow-roll parameters.*

In slow roll inflation, $\phi_{,t}$ has opposite sign to $V'(\phi)$: thus the field rolls down toward the minimum of the potential. As it does so, the universe expands. Eventually if the potential bottoms out at zero, ϵ and η grow and at some point are no longer $\ll 1$: then inflation ends, at some value $\phi_\star$. There is a number of remaining e-folds in inflation:

$$N = \int_\phi^{\phi_\star} H\,dt = -\int_{\phi_\star}^{\phi} H\frac{d\phi}{\phi_{,t}} = \int_{\phi_\star}^{\phi}\frac{3H^2}{V'(\phi)}\,d\phi = \int_{\phi_\star}^{\phi}\frac{8\pi V(\phi)}{V'(\phi)}\,d\phi. \tag{21}$$

A. Example: massive scalar field

A simple example of scalar field inflation – and one consistent with the extant data on perturbations – is provided by the potential:

$$V(\phi) = \frac{1}{2}m^2\phi^2. \tag{22}$$

This corresponds to a scalar field (spin 0 particle) with mass m. In this model, the slow-roll parameters are

$$\epsilon_{\rm V} = \frac{1}{4\pi\phi^2} \quad \text{and} \quad \eta_{\rm V} = \frac{1}{4\pi\phi^2}. \tag{23}$$

Slow roll inflation is possible for large field values; it ends when $\phi \sim \phi_\star = 1/\sqrt{4\pi}$. The number of e-folds of inflation that occur during slow roll is given by Eq. (21):

$$N = \int_{\phi_\star}^{\phi} 4\pi\phi\,d\phi = 2\pi\phi^2 - \frac{1}{2} \approx 2\pi\phi^2. \tag{24}$$

(The $-\frac{1}{2}$ is not reliable anyway because it is computed at the end of validity of slow roll.) Thus at least classically, this model is capable of generating an arbitrarily large amount of inflation if ϕ started at a sufficiently large value.