Lecture XXXVI: Introduction to cosmological perturbations

Christopher M. Hirata
*Caltech M/C 350-17, Pasadena CA 91125, USA**
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I. OVERVIEW

Reading: the subject is not covered at all in MTW, since most of the development is more recent. My principal recommendations for more in-depth reading on the subject of cosmological perturbations is Dodelson, *Modern Cosmology* (2003). However none of the standard references develop the subject from the Hamiltonian perspective as done here.

II. COSMOLOGICAL PERTURBATIONS IN FOURIER SPACE

We begin by considering a spatially flat FRW universe, which has background metric

$$ds^2 = a^2(\eta)(-d\eta^2 + \delta_{ij}dx^i dx^j). \tag{1}$$

Note here that we use the conformal time η instead of t as our time coordinate, for consistency with the rest of cosmological perturbation theory. In this case, the spatial metric is $\gamma_{ij}(\eta, x^k) = a^2(\eta)\delta_{ij}$. We will also introduce the conformal Hubble rate,

$$\mathcal{H}(\eta) = \frac{d\ln a(\eta)}{d\eta} = aH \tag{2}$$

in terms of the conventional Hubble rate H. Here we will only use $\mathcal{H}$ to avoid confusion with the Hamiltonian.

A. The Fourier space transformation

Our first step here is to try to generalize this to small spatial perturbations, which we will do in Fourier space. In general we have

$$\gamma_{ij}(\eta, x^n) = a^2(\eta)\left[\delta_{ij} + \int h_{ij}(\eta, k_n)e^{ik_m x^m}\frac{d^3k}{(2\pi)^3}\right]. \tag{3}$$

We have the restriction $h_{ij}(k_n) = h^*_{ij}(-k_n)$ in order for the spatial metric to be real. A similar transformation of the lapse and shift is possible:

$$\alpha(\eta, x^n) = a(\eta)\left[1 + \int A(\eta, k_n)e^{ik_m x^m}\frac{d^3k}{(2\pi)^3}\right] \tag{4}$$

and

$$N_i(\eta, x^n) = a^2(\eta)\int B_i(\eta, k_n)e^{ik_m x^m}\frac{d^3k}{(2\pi)^3}. \tag{5}$$

There is of course an inverse transform:

$$h_{ij}(\eta, k_n) = \int \left[a^{-2}(\eta)\gamma_{ij}(\eta, x^n) - \delta_{ij}\right] e^{-ik_m x^m}\, d^3x. \tag{6}$$

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*Electronic address: `chirata@tapir.caltech.edu`

As always, when one transforms the coordinates it is necessary to also transform the conjugate momenta so as to make the transformation canonical. Furthermore, if the transformation is time-dependent, the Hamiltonian is changed as well. In our case, the type of transformation we need is a time-dependent point transformation, which in general can be written as

$$Q^I = Q^I(q^J,\eta), \qquad p_I = \left.\frac{\partial Q^J}{\partial q^I}\right|_\eta P_J, \qquad \text{and} \qquad H_{\rm new} = H_{\rm old} + P_I \left.\frac{\partial Q^I}{\partial \eta}\right|_q . \tag{7}$$

In the above problem, the conjugate momenta to A and B_i remain zero, but the conjugate momenta to h_{ij} (which we will denote ϑ^{ij}) are now

$$\vartheta^{ij}(\eta,k_n) = \int \frac{\delta\gamma_{kl}(\eta,x^m)}{\delta h_{ij}(\eta,k_n)}\Pi^{kl}(\eta,x^m)\,d^3x = a^2(\eta)\int \Pi^{ij}(\eta,x^n)\,e^{ik_mx^m}\,\frac{d^3x}{(2\pi)^3}. \tag{8}$$

The inverse transformation is simply

$$\Pi^{ij}(\eta,x^n) = a^{-2}(\eta)\int \vartheta^{ij}(\eta,k_n)\,e^{-ik_mx^m}\,d^3k. \tag{9}$$

Note the $-$ sign in the complex exponential: the wave vector $\boldsymbol{k}$ mode of the spatial metric is canonically conjugate to the $-\boldsymbol{k}$ mode of the extrinsic curvature.

Finally, we investigate the change in the Hamiltonian. If we fix γ_{ij}, then from Eq. (6):

$$\frac{\partial h_{ij}(\eta,k_n)}{\partial\eta} = -2a^{-2}(\eta)\mathcal{H}(\eta)\int \gamma_{ij}(\eta,x^n)e^{-ik_mx^m}\,d^3x = -2\mathcal{H}(\eta)h_{ij}(\eta,k_n) + 2(2\pi)^3\mathcal{H}(\eta)\delta^{(3)}(k_n). \tag{10}$$

Then the correction to the Hamiltonian is

$$\Delta H_1 = \int \vartheta^{ij}(\eta,k_n)\left.\frac{\partial h_{ij}(\eta,k_n)}{\partial\eta}\right|_\gamma d^3k_n = -2\mathcal{H}(\eta)\int \vartheta^{ij}(\eta,k_n)h_{ij}(\eta,k_n)\,d^3k_n + 2(2\pi)^3\mathcal{H}(\eta)\vartheta^{ij}(\eta,\mathbf{0}). \tag{11}$$

B. Removal of the zero-mode

The above system has one drawback: the conjugate momentum to our perturbation $\vartheta^{ij}(\eta,k_n)$, is not zero in the background spacetime. Instead, the conjugate momentum is

$$\vartheta^{ij}(\eta,k_n)\big|_{\rm bkgnd} = a^2(\eta)\delta^{(3)}(k_n)\Pi^{ij}|_{\rm bkgnd} = \frac{-a^2\mathcal{H}(\eta)}{8\pi}\delta^{ij}\delta^{(3)}(k_n). \tag{12}$$

This is somewhat awkward since we want a conjugate momentum that is a perturbation variable as well. The solution is to do a canonical transformation that simply shifts the momentum,

$$\kappa^{ij}(\eta,k_n) = \vartheta^{ij}(\eta,k_n) + \frac{a^2\mathcal{H}(\eta)}{8\pi}\delta^{ij}\delta^{(3)}(k_n). \tag{13}$$

This comes with an associated change to the Hamiltonian,

$$\Delta H_2 = -\frac{[a^2\mathcal{H}]^\cdot}{8\pi}\delta^{ij}h_{ij}(\eta,\mathbf{0}). \tag{14}$$

We now have a system of canonically conjugate variables with which to do perturbation theory.

C. The GR Hamiltonian

To finish the job, we must re-cast the original Hamiltonian in terms of h_{ij} and κ^{ij}. In doing so, we will work only to **second order** in the perturbations, as this is what is required to get the linear perturbations. We find that

$$\int \alpha G_{ijkl}\Pi^{ij}\Pi^{kl}(\eta,x^n)\,\gamma^{-1/2}\,d^3x \;=\; a^2\int\left[1+\int A(k)e^{ikx}\frac{d^3k}{(2\pi)^3}\right]$$

$$\times\Big\{\delta_{ik}\delta_{jl}-\frac{1}{2}\delta_{ij}\delta_{kl}+2\int[\delta_{jl}h_{ik}(k)-\frac{1}{2}\delta_{kl}h_{ij}(k)]e^{ikx}\frac{d^3k}{(2\pi)^3}$$
$$+\int[h_{ik}(k)h_{jl}(k')-\frac{1}{2}h_{ij}(k)h_{kl}(k')]e^{i(k+k')x}\frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\Big\}$$
$$\times\left[-\frac{\mathcal{H}}{8\pi}\delta^{ij}+a^{-2}\int\kappa^{ij}(-k)e^{-ikx}d^3k\right]\left[-\frac{\mathcal{H}}{8\pi}\delta^{kl}+a^{-2}\int\kappa^{kl}(-k)e^{-ikx}d^3k\right]$$
$$\times\Big\{1-\frac{1}{2}\int h_{mm}(k)e^{ikx}\frac{d^3k}{(2\pi)^3}$$
$$+\frac{1}{8}\int[h_{mm}(k)h_{nn}(k')+2h_{mn}(k)h_{mn}(k')]e^{i(k+k')x}\frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\Big\}d^3x. \tag{15}$$

Here we have used the rule that, to second order in h,

$$\det(\delta_{ij}+h_{ij})=1+h_{ii}+\frac{1}{2}(h_{ii}h_{jj}-h_{ij}h_{ij})+... \tag{16}$$

and hence

$$\det(\delta_{ij}+h_{ij})^{-1/2}=1-\frac{1}{2}h_{ii}+\frac{1}{8}(h_{ii}h_{jj}+2h_{ij}h_{ij})+...\ . \tag{17}$$

We can simplify Eq. (15) by performing the x-integral; adding up the explicit terms quadratic in the perturbations, and all products of linear terms, gives:

$$\begin{aligned}\int\alpha G_{ijkl}\Pi^{ij}\Pi^{kl}(\eta,x^n)\,\gamma^{-1/2}\,d^3x \;=\;& \frac{a^2\mathcal{H}^2}{512\pi^2}\int[5h_{ij}(k)h_{ij}(-k)-3h_{ii}(k)h_{jj}(-k)]\frac{d^3k}{(2\pi)^3}\\ &+\frac{\mathcal{H}}{16\pi}\int[h_{ii}(k)\kappa^{jj}(k)-2h_{ij}(k)\kappa^{ij}(k)]d^3k\\ &+a^{-2}\int[\kappa^{ij}(k)\kappa^{ij}(-k)-\frac{1}{2}\kappa^{ii}(k)\kappa^{jj}(-k)](2\pi)^3d^3k\\ &-\frac{a^2\mathcal{H}^2}{256\pi^2}\int A(k)h_{ii}(-k)\frac{d^3k}{(2\pi)^3}+\frac{\mathcal{H}}{8\pi}\int A(k)\kappa^{ii}(k)d^3k\\ &+[\text{zero mode}],\end{aligned} \tag{18}$$

where "zero mode" refers to terms that only involve the $k=0$ Fourier mode.

For the Ricci scalar term, we recall that

$$\int\alpha\,{}^{(3)}R\gamma^{1/2}\,d^3x \tag{19}$$

has terms with no dependence on A and terms that depend on A. For the latter, we recall that to linear order

$${}^{(3)}R=a^{-2}(h_{ij,ij}-h_{ii,jj}), \tag{20}$$

and so we can see that the A-containing terms are

$$a^2\int A(k)[-k_ik_jh_{ij}(-k)+k^2h_{ii}(-k)]\,\frac{d^3k}{(2\pi)^3}. \tag{21}$$

For the terms with no A-dependence, we see that these equal $a\int{}^{(3)}R\gamma^{1/2}\,d^3x$. Now the variation of this integral is $-a\int{}^{(3)}G_{ij}\delta\gamma_{ij}\sqrt{\gamma}\,d^3x$, which vanishes for a spatially flat 3-manifold, so it follows that $a\int{}^{(3)}R\gamma^{1/2}\,d^3x$ has only quadratic (and higher) terms in h_{ij}. Furthermore, a quadratic term in the Taylor expansion in $\gamma_{ij}-a^2\delta_{ij}$ is equal to half the (functional) derivative times the perturbation. These must be given by

$$\begin{aligned}a\int{}^{(3)}R\gamma^{1/2}\,d^3x \;&=\; -\frac{1}{2}a\int{}^{(3)}G^{ij}\delta\gamma_{ij}\sqrt{\gamma}\,d^3x+\mathcal{O}(h^3)\\ &=\; -\frac{1}{4}a^2\int(2h_{k(i,j)k}-h_{ij,kk}-h_{kk,ij}-\delta_{ij}h_{kl,kl}+\delta_{ij}h_{kk,ll})h_{ij}\,d^3x\\ &=\; \frac{1}{4}a^2\int[2k_jk_kh_{ik}(k)h_{jk}(-k)-k^2h_{ij}(k)h_{ij}(-k)-2k_ik_jh_{kk}(k)h_{ij}(-k)+k^2h_{ii}(k)h_{jj}(-k)]\frac{d^3k}{(2\pi)^3}.\end{aligned} \tag{22}$$

Finally, we need the terms involving the shift. These are given by

$$
\begin{aligned}
2\int \Pi^{ij} N_{i|j}\, d^3x &= 2\int [\Pi^{ij} N_{i,j} - {}^{(3)}\Gamma^k_{ij} N_k \Pi^{ij}]\, d^3x \\
&= \int [-2\Pi^{ij}{}_{,j} N_i - \gamma^{kl}(2\gamma_{il,j} - \gamma_{ij,l}) N_k \Pi^{ij}]\, d^3x \\
&= \int [-2\Pi^{ij}{}_{,j} N_i + \frac{\mathcal{H}}{8\pi} a^{-2} \delta^{kl}(2\gamma_{il,j} - \gamma_{ij,l}) N_k \delta^{ij}]\, d^3x,
\end{aligned} \tag{23}
$$

where in the last term in the last line we worked only to second order in perturbation theory and replaced Π^{ij} by its background value. Writing this in terms of Fourier modes yields

$$
2\int \Pi^{ij} N_{i|j}\, d^3x = i\int \left[-2k_i(2\pi)^3\kappa^{ij}(-k) + \frac{\mathcal{H}}{4\pi} a^2 k_i h_{ji}(k) - \frac{\mathcal{H}}{8\pi} a^2 k_j h_{ii}\right] B_j(-k)\, \frac{d^3k}{(2\pi)^3}. \tag{24}
$$

Adding together all the GR terms, plus the corrections $\Delta H_1 + \Delta H_2$, and throwing out the zero modes gives

$$
\begin{aligned}
H_{\rm GR} + \Delta H =& \frac{a^2\mathcal{H}^2}{32\pi}\int [5h_{ij}(k)h_{ij}(-k) - 3h_{ii}(k)h_{jj}(-k)]\frac{d^3k}{(2\pi)^3} \\
&+\mathcal{H}\int [h_{ii}(k)\kappa^{jj}(k) - 4h_{ij}(k)\kappa^{ij}(k)] d^3k \\
&+16\pi a^{-2}\int [\kappa^{ij}(k)\kappa^{ij}(-k) - \frac{1}{2}\kappa^{ii}(k)\kappa^{jj}(-k)](2\pi)^3\, d^3k \\
&-\frac{a^2}{64\pi}\int [2k_ik_jh_{ik}(k)h_{jk}(-k) - k^2h_{ij}(k)h_{ij}(-k) - 2k_ik_jh_{kk}(k)h_{ij}(-k) + k^2h_{ii}(k)h_{jj}(-k)]\frac{d^3k}{(2\pi)^3} \\
&+\int \left[-\frac{a^2\mathcal{H}^2}{16\pi}h_{ii}(k) + 2\mathcal{H}(2\pi)^3\kappa^{ii}(-k) + \frac{a^2}{16\pi}k_ik_jh_{ij}(k) - \frac{a^2}{16\pi}k^2h_{ii}(k)\right] A(-k)\frac{d^3k}{(2\pi)^3} \\
&+i\int \left[-2k_i(2\pi)^3\kappa^{ij}(-k) + \frac{\mathcal{H}}{4\pi}a^2k_ih_{ji}(k) - \frac{\mathcal{H}}{8\pi}a^2k_jh_{ii}(k)\right] B_j(-k)\, \frac{d^3k}{(2\pi)^3} + [\text{zero mode}].
\end{aligned} \tag{25}
$$

III. THE MATTER TERMS

We consider the matter terms only for the simplest case – that of *cold dark matter* (CDM), consisting of massive particles that only interact gravitationally and have zero velocity dispersion. It is convenient to describe such particles as having a mean number density today of $n_0 = \rho_0/\mu$ and mass μ. Then the Lagrangian describing these particles is

$$
L = -\mu n_0 \int \frac{d\tau}{d\eta} d^3x = -\rho_0 \int \frac{d\tau}{d\eta} d^3x, \tag{26}
$$

where $d\tau$ is the proper time element for particles at Lagrangian position x. If the particles have Lagrangian coordinate displacement $\xi^i(\eta, x^k)$, then the proper time elapsed is

$$
\begin{aligned}
\frac{d\tau}{d\eta} &= \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\eta}\frac{dx^\nu}{d\eta}} \\
&= a\sqrt{[1 + A(x^k+\xi^k)]^2 - [B_i(x^k)]^2 + 2B_i(x^k+\xi^k)\dot\xi^i - [\delta_{ij} + h_{ij}(x^k+\xi^k)]\dot\xi^i\dot\xi^j} \\
&\approx a\sqrt{1 + 2A(x^k) + [A(x^k)]^2 + 2A_{,i}(x^k)\xi^i - [B_i(x^k)]^2 + 2B_i(x^k)\dot\xi^i - \dot\xi^i\dot\xi^i} \\
&\approx a\left\{1 + A(x^k) + A_{,i}(x^k)\xi^i - \frac{1}{2}[B_i(x^k)]^2 + B_i(x^k)\dot\xi^i - \frac{1}{2}\dot\xi^i\dot\xi^i\right\}.
\end{aligned} \tag{27}
$$

That is,

$$
L_{\rm CDM} = -\rho_0 a \int \left\{1 + A(x^k) + A_{,i}(x^k)\xi^i - \frac{1}{2}B_i(x^k)B_i(x^k) + B_i(x^k)\dot\xi^i - \frac{1}{2}\dot\xi^i\dot\xi^i\right\} d^3x, \tag{28}
$$

or – in Fourier space –

$$L_{\rm CDM} = -\rho_0 a \int \left\{ ik_i A(k)\xi^i(-k) - \frac{1}{2}B_i(k)B_i(-k) + B_i(k)\dot\xi^i(-k) - \frac{1}{2}\dot\xi^i(k)\dot\xi^i(-k) \right\} \frac{d^3k}{(2\pi)^3} + [\text{zero mode}], \tag{29}$$

where

$$\xi^i(k) = \int \xi^i(x)e^{-ikx}\,d^3x \quad \leftrightarrow \quad \xi^i(x) = \int \xi^i(k)e^{ikx}\,\frac{d^3k}{(2\pi)^3}. \tag{30}$$

This enables us to determine the conjugate momentum to the displacement field,

$$p_i(k) = \frac{\delta L}{\delta\dot\xi^i(k)} = (2\pi)^{-3}\rho_0 a[-B_i(-k) + \dot\xi^i(-k)], \tag{31}$$

implying

$$\dot\xi^i(k) = B_i(k) + \frac{(2\pi)^3}{\rho_0 a}p_i(-k). \tag{32}$$

The CDM Hamiltonian is then

$$\begin{aligned} H_{\rm CDM} &= \int p_i(k)\dot\xi^i(k)\,d^3k - L_{\rm CDM} \\ &= \int \left\{ ik_i\frac{\rho_0 a}{(2\pi)^3}A(k)\xi^i(-k) + B_i(k)p_i(k) + \frac{(2\pi)^3}{2\rho_0 a}p_i(k)p_i(-k) \right\} d^3k + [\text{zero mode}]. \end{aligned} \tag{33}$$

Finally, if there is a cosmological constant, an additional contribution to the Lagrangian of $-\Lambda/8\pi$ times the 4-volume per unit time arises. This is

$$\begin{aligned} L_\Lambda &= -\frac{\Lambda}{8\pi}\int \alpha\gamma^{1/2}\,d^3x \\ &= -\frac{\Lambda}{8\pi}a^4\int [1+A(x)]\left\{1 + \frac{1}{2}h_{ii}(x) + \frac{1}{8}[h_{ii}(x)h_{jj}(x) - 2h_{ij}(x)h_{ij}(x)]\right\} d^3x \\ &= -\frac{\Lambda}{8\pi}a^4\int \left[\frac{1}{2}A(k)h_{ii}(-k) + \frac{1}{8}h_{ii}(k)h_{jj}(-k) - \frac{1}{4}h_{ij}(k)h_{ij}(-k)\right] \frac{d^3k}{(2\pi)^3}. \end{aligned} \tag{34}$$

This has no effect whatsoever on the conjugate momenta, but it leads to an additional Hamiltonian that is $-L_\Lambda$:

$$H_\Lambda = \frac{\Lambda}{8\pi}a^4\int \left[\frac{1}{2}A(k)h_{ii}(-k) + \frac{1}{8}h_{ii}(k)h_{jj}(-k) - \frac{1}{4}h_{ij}(k)h_{ij}(-k)\right] \frac{d^3k}{(2\pi)^3}. \tag{35}$$

The total Hamiltonian is of course $H = H_{\rm GR} + H_\Lambda + H_{\rm CDM}$.

IV. SCALAR-VECTOR-TENSOR DECOMPOSITION

Our next order of business is to solve for the behavior of the perturbations in a ΛCDM universe – i.e. one consisting principally of cold dark matter, possibly with a cosmological constant, and spatially flat geometry. This is of direct interest since it is the universe we appear to actually inhabit!

Our lives will be made easier by the use of the symmetries of the flat FRW spacetime. First, we can see that each Fourier mode will evolve independently: to be explicit, all terms in the Hamiltonian are integrals of the form

$$\int [\text{coordinate}(k)\,\text{coordinate}(-k) + \text{coordinate}(k)\,\text{momentum}(k) + \text{momentum}(k)\,\text{momentum}(-k)]\,d^3k. \tag{36}$$

Inspection then shows that the coordinates in Fourier mode k and the conjugate momenta of Fourier mode $-k$ form a closed system. They may therefore be considered separately from each other Fourier mode. This is, of course, simply a consequence of spatial homogeneity: the Fourier modes are the irreducible representations of the group of displacements $\mathbb{R}^3$ (under vector addition).

A second simplification concerns the isotropy of the universe. Clearly this means that the behavior of the Fourier mode k depends only on its magnitude and not its direction: therefore we may consider k to point along the 3-axis. But we may make an even stronger statement about the nature of the perturbations. They can be decomposed according to their spin around the k-axis, which depends on how they transform under rotations (and reflections through a plane containing the 3-axis). For example, if we take the displacement field ξ^i, we find that under a rotation by angle Ψ around the 3-axis:

$$\begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix}_{\rm new} = \begin{pmatrix} \cos\Psi & \sin\Psi \\ -\sin\Psi & \cos\Psi \end{pmatrix} \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix}_{\rm old} \quad \text{and} \quad \xi^3_{\rm new} = \xi^3_{\rm old}. \tag{37}$$

We therefore say that ξ^1 and ξ^2 form a *spin-1* or *vector* perturbation, whereas ξ^3 forms a *spin-0* or *scalar* perturbation. **The terminology reflects the behavior under these rotations and does not imply that the objects in question are 3-vectors or 4-vectors!**

For the case of the metric perturbations, we can decompose h_{ij} in a similar way:

$$h_{ij} = \begin{pmatrix} h_{\rm T} + h_+ & h_\times & h_1 \\ h_\times & h_{\rm T} - h_+ & h_2 \\ h_1 & h_2 & h_{\rm L} \end{pmatrix} \tag{38}$$

and

$$\kappa^{ij} = \begin{pmatrix} \kappa_{\rm T} + \kappa_+ & \kappa_\times & \kappa_1 \\ \kappa_\times & \kappa_{\rm T} - \kappa_+ & \kappa_2 \\ \kappa_1 & \kappa_2 & \kappa_{\rm L} \end{pmatrix}. \tag{39}$$

Due to the repeated nature of some of the components, the $+$, $\times$, 1, 2, and T components have factors of $\frac{1}{2}$:

$$\dot{h}_+(k) = \frac{1}{2}\frac{\delta H}{\delta\kappa_+(k)}, \tag{40}$$

whereas no such factor applies to the L components. Here the L and T components are scalar perturbations, the 1 and 2 components form a vector perturbation, and the $+$ and $\times$ components form a *spin-2* or *tensor* perturbation, since they transform as:

$$\begin{pmatrix} h_+ \\ h_\times \end{pmatrix}_{\rm new} = \begin{pmatrix} \cos 2\Psi & \sin 2\Psi \\ -\sin 2\Psi & \cos 2\Psi \end{pmatrix} \begin{pmatrix} h_+ \\ h_\times \end{pmatrix}_{\rm old}. \tag{41}$$

The requirement of invariance of the Hamiltonian under rotations around the 3-axis implies that a quadratic Hamiltonian only contains terms that are quadratic in the scalars, quadratic in the vectors, and quadratic in the tensors. Cross-terms, e.g. scalar times vector, are not invariant and are not allowed. Furthermore, reflection symmetry across the 13-plane forbids terms that mix 1 and 2 components (e.g. $h_1\xi^2$) or that mix $+$ and $\times$ components (e.g. $h_+h_\times$); and invariance under rotation around the 3-axis by π/m implies that the "1 component" and "2 component" will have the same contribution to the Hamiltonian.

The rather dramatic consequence of this is that:

- The scalar, vector, and tensor perturbations evolve independently of each other.
- The two components of the vector perturbations (1 and 2) obey the same equations of motion and are independent of each other.
- The two components of the tensor perturbations ($+$ and $\times$) obey the same equations of motion and are independent of each other.

Well, as always in physics, symmetry makes life a lot easier.

In cosmologies with more general types of matter, higher spin perturbations are legal – e.g. in the case of the CMB, the octopole moment of the temperature anisotropy has a spin-3 piece, and the order-ℓ multipole has pieces up to spin ℓ. However, in linear perturbation theory (quadratic Hamiltonians), primordial density or metric perturbations can only source the scalar, vector, and tensor perturbations in the CMB. Higher order multipoles could be sourced by nonlinear interactions (i.e. cubic terms in the perturbation Hamiltonian such as that giving rise to gravitational lensing of the CMB), which we won't study.

We will now study the types of perturbations in order from easiest (tensors) to hardest (scalars).

V. TENSOR PERTURBATIONS

We consider first the tensor perturbations as these are easiest to analyze. The CDM plays no direct role in these (all it does is affect the **background** on which the perturbations play out). The tensors come in two polarizations, + and ×, which are equivalent – we will consider here the + perturbation. Essentially we are writing

$$h_{ij}(k) = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{42}$$

with k along the z-axis, and

$$\kappa^{ij}(k) = \begin{pmatrix} \kappa_+ & \kappa_\times & 0 \\ \kappa_\times & -\kappa_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{43}$$

In this case, the + polarization contribution from Eq. (25) is

$$\begin{aligned} H_{\rm GR}^{\rm tensor} &= \int \Big[\frac{a^2k^2}{32\pi}(2\pi)^{-3}h_+(k)h_+(-k) + 32\pi a^{-2}(2\pi)^3\kappa_+(k)\kappa_+(-k) - 8\mathcal{H}h_+(k)\kappa_+(k) \\ &\quad + \frac{5a^2\mathcal{H}^2 - a^4\Lambda}{16\pi}(2\pi)^{-3}h_+(k)h_+(-k)\Big]\, d^3k + [+ \leftrightarrow \times]. \end{aligned} \tag{44}$$

The equations of motion for such a perturbation can now be written down:

$$\dot h_+(k) = \frac{1}{2}\frac{\delta H_{\rm GR}^{\rm tensor}}{\delta\kappa_+(k)} = 32\pi a^{-2}(2\pi)^3\kappa_+(-k) - 4\mathcal{H}h_+(k). \tag{45}$$

and

$$\dot\kappa_+(k) = -\frac{1}{2}\frac{\delta H_{\rm GR}^{\rm tensor}}{\delta h_+(k)} = \left[-\frac{a^2k^2}{32\pi} - \frac{5a^2\mathcal{H}^2 - a^4\Lambda}{16\pi}\right](2\pi)^{-3}h_+(-k) + 4\mathcal{H}\kappa_+(k). \tag{46}$$

Then

$$\begin{aligned} \ddot h_+(k) &= 32\pi a^{-2}(2\pi)^3[\dot\kappa_+(-k) - 2\mathcal{H}\kappa_+(-k)] - 4\mathcal{H}\dot h_+(k) - 4\dot{\mathcal{H}}h_+(k) \\ &= -k^2h_+(k) - 10\mathcal{H}^2h_+(k) + 2a^2\Lambda h_+(k) + 64\pi a^{-2}(2\pi)^3\mathcal{H}\kappa_+(-k) - 4\mathcal{H}\dot h_+(k) - 4\dot{\mathcal{H}}h_+(k) \\ &= -k^2h_+(k) - 10\mathcal{H}^2h_+(k) + 2a^2\Lambda h_+(k) + 2\mathcal{H}[\dot h_+(-k) + 4\mathcal{H}h_+(k)] - 4\mathcal{H}\dot h_+(k) - 4\dot{\mathcal{H}}h_+(k) \\ &= -k^2h_+(k) - 2\mathcal{H}\dot h_+(k) - 2(\mathcal{H}^2 - a^2\Lambda)h_+(k) - 4\dot{\mathcal{H}}h_+(k). \end{aligned} \tag{47}$$

This can be simplified using the Friedmann equation for the Hubble rate. We take the form

$$\mathcal{H}^2 = a^2H^2 = \frac{8}{3}\pi\rho_0a^{-1} + \frac{\Lambda}{3}a^2. \tag{48}$$

Solving for ρ_0 gives

$$8\pi\rho_0 = 3a\mathcal{H}^2 - \Lambda a^3. \tag{49}$$

Taking the derivative of both sides and setting the left-hand side to $\dot\rho_0 = 0$ (and using $\dot a = a\mathcal{H}$ on the right-hand side) gives

$$0 = 6a\mathcal{H}\dot{\mathcal{H}} + 3a\mathcal{H}^3 - 3a^3\Lambda\mathcal{H}. \tag{50}$$

This establishes that the coefficient of h_+ on the right-hand side of Eq. (47) is simply $-k^2$, and thus in a ΛCDM universe:

$$\ddot h_+(k) + 2\mathcal{H}\dot h_+(k) + k^2h_+(k) = 0. \tag{51}$$

It is worth noting that A and B_i make no appearance here: that is because they contain 2 scalars (A and B_3) and a vector pair (B_1, B_2) and as such have no tensor components. We conclude that the gauge degrees of freedom have no effect on the tensor modes: in linear perturbation theory, h_+ and $h_\times$ are *gauge-invariant*.

We are finally interested in the solution of Eq. (51). In the case of pure matter domination (no Λ), the equation can be solved analytically; we have $\mathcal{H} = 2/\eta$ so

$$\ddot{h}_+(k) + \frac{4}{\eta}\dot{h}_+(k) + k^2 h_+(k) = 0. \tag{52}$$

The standard solution method for equations of this form is to find the behavior at $\eta \ll k^{-1}$. In this case, the last term is negligible and we have a dimensionally homogeneous equation. It has power law solutions $h_+ \propto \eta^\nu$ where

$$\nu(\nu-1) + 4\nu = 0, \tag{53}$$

so either $\nu = 0$ or $\nu = -3$. The second solution is decaying at early times and is of only theoretical interest. We then find the power series expansion,

$$h_+ = h_{+,0}\sum_{j=0}^{\infty} c_j (k\eta)^j \tag{54}$$

with $c_0 = 1$. Equation (52) gives, for the order η^{j-2} term,

$$j(j+3)c_j + c_{j-2} = 0, \tag{55}$$

so we see that all the odd c_j vanish, and by induction:

$$c_{2n} = \frac{(-1)^n}{(2n)(2n+3)\cdot(2n-2)(2n+1)\cdot\ldots\cdot(2)(5)} = 3(2n+2)\,\frac{(-1)^n}{(2n+3)!}. \tag{56}$$

Using the rule that in a power series multiplying by the exponent is equivalent to the $\eta\,\partial/\partial\eta$, and the power series for the sine, we see that

$$\begin{aligned} h_+ &= h_{+,0}\,3\left(\eta\frac{\partial}{\partial\eta}+2\right)\left[-\frac{\sin k\eta}{(k\eta)^3}+\frac{1}{(k\eta)^2}\right] \\ &= -3h_{+,0}\left(\eta\frac{\partial}{\partial\eta}+2\right)\frac{\sin k\eta}{(k\eta)^3} \\ &= -3h_{+,0}\left[\frac{k\eta\cos k\eta}{(k\eta)^3}-\frac{3k\eta\sin k\eta}{(k\eta)^4}+2\frac{\sin k\eta}{(k\eta)^3}\right] \\ &= 3h_{+,0}\frac{\sin k\eta - k\eta\cos k\eta}{(k\eta)^3}. \end{aligned} \tag{57}$$

Thus, the evolution of the metric perturbation is to start at a constant value for $\eta \ll k^{-1}$: this is expected since the wavelength is much longer than the horizon size and hence the perturbation is "frozen." When $\eta \sim k^{-1}$ the perturbation amplitude begins to evolve, and at $\eta \gg k^{-1}$ it oscillates with an envelope that decays as $\eta^{-2} \propto a^{-1}$. This corresponds to a standing gravitational wave that decays adiabatically.

VI. VECTOR PERTURBATIONS

We next consider the vector perturbations, treating the 1-component of the perturbation (in this section, we will denote the polarizations by H and V for "horizontal" and "vertical" rather than 1 and 2). Then the Hamiltonian includes GR, Λ, and CDM terms:

$$\begin{aligned} H^{\rm vector} &= \int\Bigg\{\frac{5a^2\mathcal{H}^2}{16\pi}(2\pi)^{-3}h_{\rm H}(k)h_{\rm H}(-k) - 4\mathcal{H}h_{\rm H}(k)\kappa_{\rm H}(k) + \frac{16\pi}{a^2}(2\pi)^3\kappa_{\rm H}(k)\kappa_{\rm H}(-k) - \frac{\Lambda}{16\pi}a^4 h_{\rm H}(k)h_{\rm H}(-k) \\ &\quad + \frac{(2\pi)^3}{2\rho_0 a}p_{\rm H}(k)p_{\rm H}(-k) + \left[-2ik\kappa_{\rm H}(-k) + i\frac{\mathcal{H}}{4\pi}(2\pi)^{-3}a^2 k h_{\rm H}(k) + p_{\rm H}(-k)\right]B_{\rm H}(-k)\Bigg\}\,d^3k \\ &\quad + [{\rm H} \leftrightarrow {\rm V}]. \end{aligned} \tag{58}$$

The quantity in square brackets is constrained to be zero as part of the legal initial condition – that is, we have the vector constraint:

$$-2ik(2\pi)^3\kappa_{\rm H}(-k) + i\frac{\mathcal{H}}{4\pi}a^2 k h_{\rm H}(k) + p_{\rm H}(-k) = 0. \tag{59}$$

That is, given the vector part of the momentum density, one combination of the metric perturbation and the extrinsic curvature perturbation is immediately known.

We are also interested in the equations of motion. Note that $\xi_{\rm H}(k)$ makes no appearance in Eq. (58) – this corresponds to a symmetry in which the particles are re-labeled: the "reference" position of a particle (remember we are using a Lagrangian description!) can be displaced by any divergence-free field (such as the spin-1 part of a 3-vector) with no consequence to the dynamics. The corresponding conservation law is that of the conjugate momentum $p_{\rm H}(k)$. In fact $ikp_{\rm H}(k)$ is the vorticity, which is part of the more general rule that the freezing of vortex lines is the "conservation law" associated with the re-labeling symmetry.

The evolution equation for the particle displacement is simply

$$\dot{\xi}_{\rm H}(k) = \frac{\delta H^{\rm vector}}{\delta p_{\rm H}(k)} = \frac{(2\pi)^3}{\rho_0 a} p_{\rm H}(-k) + B_{\rm H}(k). \tag{60}$$

This equation makes sense: the particle displacement rate is simply associated with the momentum per unit mass, plus a term associated with how far the coordinate system moves relative to the normal observer (the shift). In fact this equation integrates easily:

$$\xi_{\rm H}(k,\eta) = \frac{(2\pi)^3}{\rho_0} p_{\rm H}(-k) \int a^{-1}\,d\eta + \int B_{\rm H}(k,\eta)\,d\eta. \tag{61}$$

The particle's physical velocity relative to the normal observer scales as a^{-1}, i.e. it decreases as the Universe expands. Thus in the ΛCDM case, this vector mode is always decaying. Of course, the (additive) time-independent constant in $\xi_{\rm H}(k,\eta)$ has no physical meaning.

We still have to figure out the evolution equation for the metric perturbations. In fact, Eq. (58) gives these – for the metric derivative, we find

$$\dot{h}_{\rm H}(k) = \frac{1}{2}\frac{\delta H^{\rm vector}}{\delta \kappa_{\rm H}(k)} = -2\mathcal{H}h_{\rm H}(k) + \frac{16\pi}{a^2}(2\pi)^3\kappa_{\rm H}(-k) - ikB_{\rm H}(k) \tag{62}$$

and hence, using the constraint equation,

$$\dot{h}_{\rm H}(k) = -8\pi i a^{-2} p_{\rm H}(-k) - ikB_{\rm H}(k). \tag{63}$$

This gives a deterministic evolution for the spatial metric perturbation; but note that as a becomes large, the first term drops to zero and the second term is pure gauge. We thus conclude that **vector perturbations decay away in an expanding universe**, except for a gauge degree of freedom that has arbitrary behavior. A simple choice of gauge, such as $B_{\rm H} = -\gamma k^{-1} h_{\rm H}$, suffices to send the vector metric perturbation to zero in the future.

It is possible to write down the evolution equation for $\kappa_{\rm H}(-k)$, but it is redundant with the constraint Eq. (59).

We conclude that vector perturbations in ΛCDM correspond to a single physical vorticity mode, which is decaying.

VII. SCALAR PERTURBATIONS

Finally, we come to the scalar perturbations. The scalar sector has more coordinates and conjugate momenta than the other sectors: the coordinates are $\xi_{\rm L} \equiv \xi^3$, $h_{\rm T}$, and $h_{\rm L}$, in addition to the nondynamical A and $B_{\rm L} \equiv B_3$.

$$\begin{aligned} H^{\rm scalar} \;=\; & \int \Big\{ \frac{a^2\mathcal{H}^2}{16\pi}(2\pi)^{-3}[-h_{\rm T}(k)h_{\rm T}(-k) + h_{\rm L}(k)h_{\rm L}(-k) - 6h_{\rm T}(k)h_{\rm L}(-k)] \\ & + \mathcal{H}[-4h_{\rm T}(k)\kappa_{\rm T}(k) + 2h_{\rm T}(k)\kappa_{\rm L}(k) + 2h_{\rm L}(k)\kappa_{\rm T}(k) - 3h_{\rm L}(k)\kappa_{\rm L}(k)] \\ & + \frac{16\pi}{a^2}(2\pi)^3\Big[-2\kappa_{\rm T}(k)\kappa_{\rm L}(-k) + \frac{1}{2}\kappa_{\rm L}(k)\kappa_{\rm L}(-k)\Big] - \frac{a^2k^2}{32\pi}h_{\rm T}(k)h_{\rm T}(-k) \\ & - \frac{\Lambda a^4}{64\pi}(2\pi)^{-3}[4h_{\rm T}(k)h_{\rm L}(-k) + h_{\rm L}(k)h_{\rm L}(-k)] + \frac{(2\pi)^3}{2\rho_0 a}p_{\rm L}(k)p_{\rm L}(-k) \\ & + (2\pi)^{-3}A(-k)\Big[-\frac{a^2(\mathcal{H}^2+k^2)}{8\pi}h_{\rm T}(k) - \frac{a^2\mathcal{H}^2}{16\pi}h_{\rm L}(k) + 2\mathcal{H}(2\pi)^3[2\kappa_{\rm T}(-k) + \kappa_{\rm L}(-k)] \\ & + \frac{\Lambda a^4}{8\pi}h_{\rm T}(k) + \frac{\Lambda a^4}{16\pi}h_{\rm L}(k) - i\rho_0 ak\xi_{\rm L}(k)\Big] \\ & + iB_{\rm L}(-k)\Big[-2k\kappa_{\rm L}(-k) - \frac{a^2\mathcal{H}}{4\pi(2\pi)^3}kh_{\rm T}(k) + \frac{a^2\mathcal{H}}{8\pi(2\pi)^3}kh_{\rm L}(k) - ip_{\rm L}(-k)\Big]\Big\}\,d^3k. \end{aligned} \tag{64}$$

The factors multiplying $A(-k)$ and $B_{\rm L}(-k)$ are constraints and must be zero.

A. Matter perturbations

We may solve the dynamics of the scalar perturbations by the standard construction of the Hamiltonian equations of motion. For the CDM, this construction gives

$$\dot{\xi}_{\rm L}(k) = \frac{\delta H^{\rm scalar}}{\delta p_{\rm L}(k)} = \frac{(2\pi)^3}{\rho_0 a} p_{\rm L}(-k) + B_{\rm L}(k) \tag{65}$$

for the displacement and

$$\dot{p}_{\rm L}(-k) = -\frac{\delta H^{\rm scalar}}{\delta \xi_{\rm L}(-k)} = -\frac{i\rho_0 a k}{(2\pi)^3} A(k) \tag{66}$$

for the momentum.

B. Metric perturbations

The spatial metric perturbations $h_{\rm L}$ and $h_{\rm T}$ also obey evolution equations. These are:

$$\dot{h}_{\rm L}(k) = \frac{\delta H^{\rm scalar}}{\delta \kappa_{\rm L}(k)} = \mathcal{H}[2h_{\rm T}(k) - 3h_{\rm L}(k)] + \frac{16\pi}{a^2}(2\pi)^3[-2\kappa_{\rm T}(-k) + \kappa_{\rm L}(-k)] + 2\mathcal{H}A(k) - 2ikB_{\rm L}(k) \tag{67}$$

for the longitudinal-scalar part and

$$\dot{h}_{\rm T}(k) = \frac{1}{2}\frac{\delta H^{\rm scalar}}{\delta \kappa_{\rm T}(k)} = \mathcal{H}[-2h_{\rm T}(k) + h_{\rm L}(k)] - \frac{16\pi}{a^2}(2\pi)^3 \kappa_{\rm L}(-k) + 2\mathcal{H}A(k) \tag{68}$$

for the transverse-scalar part.

It is possible to work out the evolution equations for $\kappa_{\rm T}(k)$ and $\kappa_{\rm L}(k)$ separately, but it is easier to use the constraints [the terms multiplying A and B in Eq. (64)] to solve for them. The $B_{\rm L}$ constraint gives

$$\kappa_{\rm L}(-k) = \frac{a^2\mathcal{H}}{16\pi(2\pi)^3}[h_{\rm L}(k) - 2h_{\rm T}(k)] + \frac{i}{2k}p_{\rm L}(-k), \tag{69}$$

and the A constraint gives

$$2\kappa_{\rm T}(-k) + \kappa_{\rm L}(-k) = \frac{a^2}{32\pi\mathcal{H}(2\pi)^3}[2(\mathcal{H}^2 + k^2 - \Lambda a^2)h_{\rm T}(k) + (\mathcal{H}^2 - \Lambda a^2)h_{\rm L}(k)] + \frac{i\rho_0 a k}{2\mathcal{H}(2\pi)^3}\xi_{\rm L}(k). \tag{70}$$

This implies

$$\kappa_{\rm T}(-k) = \frac{a^2}{64\pi\mathcal{H}(2\pi)^3}[2(3\mathcal{H}^2 + k^2 - \Lambda a^2)h_{\rm T}(k) + (-\mathcal{H}^2 - \Lambda a^2)h_{\rm L}(k)] + \frac{i\rho_0 a k}{4\mathcal{H}(2\pi)^3}\xi_{\rm L}(k) - \frac{i}{4k}p_{\rm L}(-k). \tag{71}$$

Equations (69) and (71) provide us the conjugate momenta to the metric without having to follow additional evolution equations. We may use them to transform Eq. (68) into

$$\dot{h}_{\rm T}(k) = -i\frac{8\pi}{a^2 k}(2\pi)^3 p_{\rm L}(-k) + 2\mathcal{H}A(k). \tag{72}$$

The longitudinal part also simplifies but with a bit more work:

$$\dot{h}_{\rm L}(k) = -\frac{1}{2\mathcal{H}}[2(3\mathcal{H}^2 + k^2 - \Lambda a^2)h_{\rm T}(k) + (3\mathcal{H}^2 - \Lambda a^2)h_{\rm L}(k)] - \frac{8\pi i\rho_0 k}{a\mathcal{H}}\xi_{\rm L}(k) + \frac{16\pi i}{ka^2}(2\pi)^3 p_{\rm L}(-k) + 2\mathcal{H}A(k) - 2ikB_{\rm L}(k). \tag{73}$$

This can be put in its conventional form using the Friedmann equation, $3\mathcal{H}^2 - \Lambda a^2 = 8\pi\rho_0 a^{-1}$. Then:

$$\dot{h}_{\rm L}(k) = -\frac{k^2}{\mathcal{H}}h_{\rm T}(k) - \frac{4\pi\rho_0}{a\mathcal{H}}[2h_{\rm T}(k) + h_{\rm L}(k) + 2ik\xi_{\rm L}(k)] + \frac{16\pi i}{ka^2}(2\pi)^3 p_{\rm L}(-k) + 2\mathcal{H}A(k) - 2ikB_{\rm L}(k). \tag{74}$$

Equations (72) and (74) allow us to describe the evolution of the spatial metric directly in terms of its "current" value and the matter fields. The first-order nature of the scalar sector is a direct consequence of the constraints, and tells us that the gravitational sector contains no scalar degrees of freedom.

C. Solution in synchronous gauge

Equations (65) and (66) are general, but inspection of them shows that given any initial configuration, one may choose a δ-function lapse

$$A(k) = -i\frac{(2\pi)^3}{\rho_0 a k} p_{\rm L}(-k, \eta_{\rm initial})\delta(\eta - \eta_{\rm initial} - \epsilon), \tag{75}$$

which sets $p_{\rm L} = 0$ for all future time. Further, if we chose $B_{\rm L}(k) = 0$ then we have constructed a gauge choice in which the particle momenta (in the normal frame) are zero and the Lagrangian displacements are constant. This gauge is called the *synchronous gauge*, and is used in almost all numerical cosmological perturbation theory codes. In synchronous gauge, the CDM particles follow trajectories of fixed spatial coordinates x^i, and these threads are normal to the slices of constant t. Note that the synchronous gauge is not unique: one can still add any constant to $\xi_{\rm L}(k)$ by inserting a δ-function in the shift. For our purposes here, we will take $\xi_{\rm L}(k)$ to be zero – this is a special choice of synchronous gauge.

In this special synchronous gauge, the evolution equations become

$$\dot{h}_{\rm T}(k) = 0 \quad \text{and} \quad \dot{h}_{\rm L}(k) = -\frac{k^2}{\mathcal{H}} h_{\rm T}(k) - \frac{4\pi\rho_0}{a\mathcal{H}}[2h_{\rm T}(k) + h_{\rm L}(k)]. \tag{76}$$

Equation (76) is a system of two first-order ODEs and has two linearly independent solutions. It is straightforward to find both. Clearly $h_{\rm T}$ is constant (independent of η): we set $h_{\rm T}(k) = 2C_1$. Then we set $h = 2h_{\rm T}(k) + h_{\rm L}(k)$, and see that

$$\dot{h} + \frac{4\pi\rho_0}{a\mathcal{H}} h = -\frac{2k^2}{\mathcal{H}} C_1. \tag{77}$$

This is a linear ODE with variable coefficients and can be solved by standard methods. We define the integrating factor

$$\mu(\eta) \equiv \int \frac{4\pi\rho_0}{a\mathcal{H}} d\eta = 4\pi\rho_0 \int \frac{1}{a^2\mathcal{H}^2} da = 12\pi\rho_0 \int \frac{1}{a(8\pi\rho_0 + \Lambda a^3)} da. \tag{78}$$

Using the rule that

$$\begin{aligned}
\int \frac{1}{a(1 + a^3/a_0^3)} da &= \frac{1}{3}\int \frac{1}{y(1+y)} dy \\
&= \frac{1}{3}\int \left(\frac{1}{y} - \frac{1}{1+y}\right) dy \\
&= \frac{1}{3}\ln\frac{y}{1+y} + \text{const} \\
&= \frac{1}{3}\ln\frac{a^3}{1 + a^3/a_0^3} + \text{const}
\end{aligned} \tag{79}$$

(where we substituted $y = a^3/a_0^3$), we may identify this with the integrand in μ for $a_0^3 = 8\pi\rho_0/\Lambda$ and set

$$e^\mu = a^{3/2}\left(1 + \frac{\Lambda a^3}{8\pi\rho_0}\right)^{-1/2}. \tag{80}$$

(Technically there could be a constant prefactor but we don't care.) Here a_0 denotes the scale factor at which the mean density of the universe is $\Lambda/8\pi$, i.e. at which the matter and Λ densities are equal. For $a \ll a_0$ the cosmological constant is irrelevant, and for $a \gg a_0$ the cosmological constant dominates.

Then

$$\frac{d}{d\eta}(e^\mu h) = e^\mu(\dot{h} + \dot{\mu}h) = -2e^\mu \frac{k^2 C_1}{\mathcal{H}}, \tag{81}$$

and so

$$h = -2k^2 C_1 e^{-\mu}\int e^\mu \mathcal{H}^{-1}\, d\eta \;=\; -2k^2 C_1 e^{-\mu}\int e^\mu \frac{1}{a\mathcal{H}^2}\, da$$

$$
\begin{aligned}
&= -\frac{k^2 C_1}{2\pi\rho_0} e^{-\mu} \int e^{\mu} a \frac{d\mu}{da}\, da \\
&= -\frac{k^2 C_1}{2\pi\rho_0} e^{-\mu} \Big[a e^{\mu} - \int e^{\mu}\, da \Big] \\
&= -\frac{k^2 C_1}{2\pi\rho_0} \Big[a - e^{-\mu} \int_0^a e^{\mu}\, da \Big] + C_2 e^{-\mu}, \qquad (82)
\end{aligned}
$$

where we have explicitly separated out the integration constant C_2.

It is conventional to define the *growth function* $G(a)$ as

$$
G(a) \equiv \frac{5}{3} \Big[a - e^{-\mu} \int_0^a e^{\mu}\, da \Big], \qquad (83)
$$

so that $h = -(3k^2 C_1/10\pi\rho_0) G(a) + C_2 e^{-\mu}$.

We thus find the overall solution:

$$
h_{\rm T}(k) = 2C_1, \quad h_{\rm L}(k) = -4C_1 - \frac{3k^2 C_1}{10\pi\rho_0} G(a) + C_2 a^{-3/2} \left(1 + \frac{\Lambda a^3}{8\pi\rho_0} \right)^{1/2}. \qquad (84)
$$

The CDM particles of course remain stationary in this coordinate system, so the density perturbation in synchronous gauge is simply the perturbation to $\det\gamma$:

$$
\frac{\delta\rho}{\bar\rho}(k)\Big|_{\rm sync} = -\frac{1}{2} h_{\rm L}(k) - h_{\rm T}(k) = \frac{3k^2 C_1}{20\pi\rho_0} G(a) - \frac{1}{2} C_2 a^{-3/2} \left(1 + \frac{\Lambda a^3}{8\pi\rho_0} \right)^{1/2}. \qquad (85)
$$

(Of course this is gauge-dependent.)

The C_2 solution diverges at early times, and so in a universe that started with small perturbations (as seen in the cosmic microwave background) we suspect it to be tiny. (There is a subtlety here: it is excited during the radiation-dominated epoch on small scales, but we aren't considering radiation yet.) As far as the growth of large scale structure, galaxy formation, etc., C_2 can be ignored. This leaves us with C_1, which is the one that could be well-behaved at early times.

D. Growth function

The growth of large scale structure with time in linear perturbation theory is thus driven by the single function $G(a)$. In cosmologies with $\Lambda = 0$, we have $e^{\mu} = a^{3/2}$ and

$$
G(a) = \frac{5}{3} \Big[a - e^{-\mu} \int_0^a e^{\mu}\, da \Big] = \frac{5}{3} \Big[a - a^{-3/2} \int_0^a a^{3/2}\, da \Big] = \frac{5}{3} \Big[a - \frac{2}{5} a \Big] = a. \qquad (86)
$$

Thus matter density perturbations grow linearly with the scale factor.

In universes with a cosmological constant (like ours?) the growth function behaves differently. There is not an analytic solution for the integral, but we may Taylor expand:

$$
\begin{aligned}
G(a) &= \frac{5}{3} \Big[a - a^{-3/2} \Big(1 + \frac{a^3}{a_0^3} \Big)^{1/2} \int_0^a a^{3/2} \Big(1 + \frac{a^3}{a_0^3} \Big)^{-1/2} da \Big] \\
&\approx \frac{5}{3} \Big[a - a^{-3/2} \Big(1 + \frac{a^3}{2a_0^3} - \frac{a^6}{8a_0^6} \Big) \int_0^a a^{3/2} \Big(1 - \frac{a^3}{2a_0^3} + \frac{3a^6}{8a_0^6} \Big) da \Big] \\
&\approx \frac{5}{3} \Big[a - a^{-3/2} \Big(1 + \frac{a^3}{2a_0^3} - \frac{a^6}{8a_0^6} \Big) a^{5/2} \Big(\frac{2}{5} - \frac{a^3}{11a_0^3} + \frac{3a^6}{68a_0^6} \Big) \Big] \\
&\approx \frac{5}{3} a \Big[1 - \Big(\frac{2}{5} + \frac{6a^3}{65a_0^3} - \frac{48a^6}{935a_0^6} \Big) \Big] \qquad (87)
\end{aligned}
$$

so

$$
G(a) = a \Big[1 - \frac{2a^3}{13a_0^3} - \frac{16a^6}{187a_0^6} + \ldots \Big]. \qquad (88)
$$

Thus we see that **the cosmological constant causes the growth of structure to proceed more slowly than in an $\Omega_m = 1$ universe**. This is the basis for several of the tests of dark energy (cluster abundance, gravitational lensing ...).

Even more interesting is the behavior of the growth function at $a \gg a_0$, i.e. in the far future. We may solve this by writing

$$\begin{aligned} G(a) &= \frac{5}{3}\int_0^a [1 - e^{-\mu(a)+\mu(a')}]\, da' \\ &= \frac{5}{3}\int_0^a \left[1 - \frac{a'^{3/2}(a_0^3 + a'^3)^{-1/2}}{a^{3/2}(a_0^3 + a^3)^{-1/2}}\right] da' \\ &= \frac{5}{3}\int_0^a \left[1 - \frac{(a_0^3 a^{-3} + 1)^{1/2}}{(a_0^3 a'^{-3} + 1)^{1/2}}\right] da'. \end{aligned} \tag{89}$$

This integral converges in the limit of $a \to \infty$: we find

$$G(a = \infty) = \frac{5}{3}\int_0^\infty \left[1 - (a_0^3 a'^{-3} + 1)^{-1/2}\right] da'. \tag{90}$$

Setting $a' = a_0 y$, this simplifies to

$$G(a = \infty) = \frac{5}{3}a_0 \int_0^\infty \left[1 - (y^{-3} + 1)^{-1/2}\right] dy, \tag{91}$$

where the integrand behaves as $\propto y^{3/2}$ at small y and $\propto y^{-3}$ at large y. Thus the growth of structure in a Λ-dominated cosmology eventually freezes: linear perturbations asymptotically approach a final state of finite density perturbation!