Lecture XXXV: Hamiltonian formulation of GR

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I. OVERVIEW

The next step is to proceed to construct the full Hamiltonian formulation of GR. Clearly this includes the Hamiltonian itself, but just as important is the canonical structure: what are the fields, what is conjugate to what, etc.

Reading:

- MTW §§21.6–21.8.

II. RECASTING THE LAGRANGIAN

The Lagrangian for GR is remarkably simple:

$$S_{\rm GR} = \frac{1}{16\pi}\int R\sqrt{-g}\,d^4x \qquad \rightarrow \qquad L = \frac{1}{16\pi}\int R\sqrt{-g}\,d^3x. \tag{1}$$

However, while the action is manifestly covariant, the construction of "initial conditions" and the time evolution development necessitate breaking manifest general covariance: we must slice the spacetime as described in the previous lecture. The task, then, will be to write the Lagrangian in terms of the metric, any spatial derivatives, and the **first** time derivative. Only then can we employ the standard construction of the Hamiltonian. In this section, we explicitly set the signature parameter to $s = -1$.

We will choose in this case as our basic variables the lapse α, the shift N_i, and the spatial metric γ_{ij} associated with slices Σ_t. All of $S_{\rm GR}$ must be expressible in terms of these objects and their time derivatives – or, equivalently, $\dot\alpha$, $\dot N_i$, and K_{ij} (since this is related to $\dot\gamma_{ij}$).

In this system, the determinant of the metric tensor is simply given by

$$\begin{aligned} g &= \left|\begin{array}{c|c} N_iN^i - \alpha^2 & N_i \\ \hline N_i & \gamma_{ij}\end{array}\right| \\ &= \left|\begin{array}{c|c} N_iN^i & \gamma_{ik}N^k \\ \hline N_i & \gamma_{ij}\end{array}\right| - \alpha^2\,|\gamma_{ij}| \\ &= 0 - \alpha^2\gamma = -\alpha^2\gamma. \end{aligned} \tag{2}$$

[In the second line, we have used the expansion of the determinant to extract the contribution from the $-\alpha^2$ term, which multiplies the determinant of the $(D-1)\times(D-1)$ matrix γ_{ij}; and then the first determinant on the second line must vanish, since the top row is equal to the linear combination of N^1 times the second row, plus N^2 times the third row ... plus N^{D-1} times the last row.] Thus

$$\sqrt{-g} = \alpha\sqrt{\gamma}. \tag{3}$$

The handling of R is harder. First, we see that in Gaussian normal coordinates,

$$R = R^{ij}{}_{ij} + R^{ti}{}_{ti} + R^{it}{}_{it} = R^{ij}{}_{ij} + 2R^{ti}{}_{ti} = {}^{(D-1)}R - K^i_jK^j_i + K^2 + 2R^{\mu\alpha}{}_{\nu\beta}n_\mu n^\nu\gamma^\beta_\alpha. \tag{4}$$

The first and last expressions depend only on the surface and not on the nature of the coordinate system off the surface, so they are equal in any general system. Most of the terms in the last expression depend only on first

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derivatives of the metric, but the Riemann tensor term is undesirable. We may recast it in a more useful form by using

$$\begin{aligned} R^{\mu\alpha}{}_{\nu\beta} n_\mu n^\nu \gamma^\beta_\alpha &= R^\alpha{}_{\mu\nu\beta} n^\mu n^\nu \gamma^\beta_\alpha \\ &= R^\alpha{}_{\mu\nu\beta} n^\mu n^\nu (\delta^\beta_\alpha + n^\beta n_\alpha) \\ &= R^\alpha{}_{\mu\nu\alpha} n^\mu n^\nu + R^\alpha{}_{\mu\nu\beta} n^\mu n^\nu n^\beta n_\alpha \\ &= R^\alpha{}_{\mu\nu\alpha} n^\mu n^\nu \\ &= n^\alpha{}_{;\alpha\nu} n^\nu - n^\alpha{}_{;\nu\alpha} n^\nu \\ &= (n^\alpha{}_{;\alpha} n^\nu)_{;\nu} - n^\alpha{}_{;\alpha} n^\nu{}_{;\nu} - (n^\alpha{}_{;\nu} n^\nu)_{;\alpha} + n^\alpha{}_{;\nu} n^\nu{}_{;\alpha}. \end{aligned} \tag{5}$$

Two of these terms are total derivatives. The others can be simplified by considering the tensor $X^\alpha{}_\nu = n^\alpha{}_{;\nu}$. The definition of extrinsic curvature gives the components $X^i{}_j = -K^i_j$, and also

$$X^\alpha{}_\nu n_\alpha = n^\alpha{}_{;\nu} n_\alpha = \frac{1}{2}(n^\alpha n_\alpha)_{;\nu} = 0, \tag{6}$$

so $X^t{}_t = 0$ and $X^t{}_i = 0$. Then

$$-n^\alpha{}_{;\alpha} n^\nu{}_{;\nu} + n^\alpha{}_{;\nu} n^\nu{}_{;\alpha} = -K^2 + K^i_j K^j_i. \tag{7}$$

Combining with Eq. (4), we see that

$$R = {}^{(D-1)}R - K^i_j K^j_i + K^2 + 2(-K^2 + K^i_j K^j_i) + \text{total derivative}, \tag{8}$$

which simplifies to

$$R = {}^{(D-1)}R - K^2 + K^i_j K^j_i + \text{total derivative}. \tag{9}$$

Thus – aside from irrelevant total derivative terms – the Lagrangian reduces to

$$L_{\rm GR} = \frac{1}{16\pi} \int \left[{}^{(D-1)}R - K^2 + K^i_j K^j_i \right] \alpha \sqrt{\gamma}\, d^3x \tag{10}$$

or

$$L_{\rm GR} = \frac{1}{16\pi} \int \left[{}^{(D-1)}R - (\gamma^{ij} K_{ij})^2 + \gamma^{ij}\gamma^{kl} K_{ik} K_{jl} \right] \alpha \sqrt{\gamma}\, d^3x. \tag{11}$$

The extrinsic curvature, which appears explicitly here, is a function of $\dot\gamma_{ij}$:

$$K_{ij} = \frac{1}{2\alpha}(-\dot\gamma_{ij} + N_{i|j} + N_{j|i}); \tag{12}$$

it is here that one finds all the dependence on $\dot\gamma_{ij}$ (so that there is dynamics!) and on the shift N_i.

III. HAMILTONIAN OF GR

Now we are finally ready to Hamiltonianize Einstein's equations. We first find the conjugate momenta, and then construct the Legendre transform.

A. Conjugate momenta

We consider here only the case where the matter Lagrangian depends only on the fields and on the metric tensor, and not on the derivatives of the metric. This allows us to build the conjugate momenta only from $L_{\rm GR}$. This is true of e.g. the swarm of particles, electromagnetic fields, and scalar fields; if it were not true then of course the conjugate momenta would be affected.

So we begin with the spatial metric: its time derivative appears only in K_{ij}, and the relation between K_{ij} and $\dot{\gamma}_{ij}$ is merely a factor of $-1/(2\alpha)$. Then the conjugate momenta to the spatial metric are

$$\Pi^{ij} = \frac{\delta L_{\rm GR}}{\delta \gamma_{ij}} = \left(-\frac{1}{8\pi}\gamma^{ij}\gamma^{kl}K_{kl} + \frac{1}{8\pi}\gamma^{ik}K_{kl}\gamma^{jl}\right)\frac{-1}{2\alpha}\,\alpha\sqrt{\gamma}, \tag{13}$$

or

$$\Pi^{ij} = \frac{1}{16\pi}(K\gamma^{ij} - K^{ij})\sqrt{\gamma}. \tag{14}$$

Note that the conjugate momentum is a $(D-1)\times(D-1)$ symmetric tensor, except for the volume factor $\sqrt{\gamma}$ associated with the functional derivative (which would not be in a standard tensor). We see that the extrinsic curvature is (aside from the trace term) conjugate to the spatial metric. It is possible to solve for the extrinsic curvature in terms of the conjugate momentum by taking the trace of Eq. (14), yielding

$$\frac{16\pi}{D-2}\gamma^{-1/2}\Pi = K, \tag{15}$$

and then

$$K^{ij} = 16\pi\gamma^{-1/2}\left(\frac{1}{D-2}\Pi\gamma^{ij} - \Pi^{ij}\right). \tag{16}$$

This will be very useful in building the Hamiltonian.

It is natural to ask what are the conjugate momenta to α and N_i, which we denote by Π_α and $[\Pi_N]^i$. Since the time derivatives of the lapse and shift do not appear in the Lagrangian, the answer is simple:

$$\Pi_\alpha = 0 \quad \text{and} \quad [\Pi_N]^i = 0. \tag{17}$$

These are the *primary constraints*. In general, constraints in Hamiltonian mechanics restrict us to subspaces of the overall phase space. They tell us that out of what appear to be 20 independently specifiable functions at each point (the 10 metric components and their conjugate momenta), in fact 4 of the conjugate momenta are not real degrees of freedom.

B. Hamiltonian

The Hamiltonian is constructed by the usual Legendre transform method. For the purely GR parts of the Hamiltonian, we have

$$H_{\rm GR} = \int (\Pi^{ij}\dot{\gamma}_{ij} + [\Pi_N]^i\dot{N}_i + \Pi_\alpha\dot{\alpha})\,d^3x - L_{\rm GR}. \tag{18}$$

We've already seen that $\Pi_\alpha = 0$ and $[\Pi_N]^i = 0$, so the integral simplifies. Moreover,

$$\dot{\gamma}_{ij} = -2\alpha K_{ij} + N_{i|j} + N_{j|i} = -32\pi\alpha\gamma^{-1/2}\left(\frac{1}{D-2}\Pi\gamma_{ij} - \Pi_{ij}\right) + N_{i|j} + N_{j|i}. \tag{19}$$

Therefore

$$\begin{aligned} H_{\rm GR} &= -32\pi\int \Pi^{ij}\left(\frac{1}{D-2}\Pi\gamma_{ij} - \Pi_{ij}\right)\alpha\gamma^{-1/2}\,d^3x + 2\int \Pi^{ij}N_{i|j}\,d^3x - L_{\rm GR} \\ &= -32\pi\int\left(\frac{1}{D-2}\Pi^2 - \Pi_{ij}\Pi^{ij}\right)\alpha\gamma^{-1/2}\,d^3x + 2\int \Pi^{ij}N_{i|j}\,d^3x - L_{\rm GR}. \end{aligned} \tag{20}$$

To finish the Hamiltonian, we must express $L_{\rm GR}$ in terms of conjugate momenta. Using Eq. (16), we have

$$\begin{aligned} -K^2 + K^i_jK^j_i &= -\frac{256\pi^2}{(D-2)^2}\gamma^{-1}\Pi^2 + 256\pi^2\gamma^{-1}\left(\frac{1}{D-2}\Pi\delta^i_j - \Pi^i_j\right)\left(\frac{1}{D-2}\Pi\delta^j_i - \Pi^j_i\right) \\ &= 256\pi^2\gamma^{-1}\left(-\frac{1}{D-2}\Pi^2 + \Pi^i_j\Pi^j_i\right). \end{aligned} \tag{21}$$

This allows us to remove the extrinsic curvature terms in $L_{\rm GR}$. Moreover, we may use integration by parts to simplify the shift term:

$$\int \Pi^{ij} N_{i|j}\, d^3x = \int \Pi^{ij}\gamma^{-1/2} N_{i|j}\,\gamma^{1/2}\, d^3x = -\int (\Pi^{ij}\gamma^{-1/2})_{|j} N_i\,\gamma^{1/2}\, d^3x. \tag{22}$$

This leads us to the final Hamiltonian:

$$H_{\rm GR} = 16\pi \int \left(-\frac{1}{D-2}\Pi^2 + \Pi_{ij}\Pi^{ij}\right) \alpha\gamma^{-1/2}\, d^3x - \frac{1}{16\pi}\int {}^{(D-1)}R\,\alpha\gamma^{1/2}\, d^3x - 2\int (\Pi^{ij}\gamma^{-1/2})_{|j} N_i\,\gamma^{1/2}\, d^3x. \tag{23}$$

Remember that there may be an additional Hamiltonian term associated with the matter fields.

C. Secondary constraints

We have now built a Hamiltonian, and we have a set of coordinates and momenta $\{\alpha, N_i, \gamma_{ij}, \Pi_\alpha, [\Pi_N]^i, \Pi^{ij}\}$. However, this does not completely solve our problem, because we do not yet know what parts of the phase space are physically allowed. We already learned that there are the primary constraints that restrict the possible range of conjugate momenta, Eq. (17). A further set of constraints – the *secondary constraints* – arise from requiring that the primary constraints remain satisfied as the system evolves. To obtain these, one writes

$$\dot\Pi_\alpha = -\frac{\delta H}{\delta\alpha} = 0 \quad \text{and} \quad [\dot\Pi_N]^i = -\frac{\delta H}{\delta N_i} = 0. \tag{24}$$

Using Eq. (23), we see that these equations imply

$$16\pi\left(-\frac{1}{D-2}\Pi^2 + \Pi_{ij}\Pi^{ij}\right)\gamma^{-1/2} - \frac{1}{16\pi}\,{}^{(D-1)}R\,\gamma^{1/2} + \frac{\delta H_{\rm matter}}{\delta\alpha} = 0 \tag{25}$$

and

$$-2(\Pi^{ij}\gamma^{-1/2})_{|j}\gamma^{1/2} + \frac{\delta H_{\rm matter}}{\delta N_i} = 0. \tag{26}$$

These are the secondary constraints. Note that they are constraints on the legal γ_{ij} and Π^{ij} – i.e. on the spatial geometry and extrinsic curvature. Thus if initial conditions are to be specified via $\{\alpha, N_i, \gamma_{ij}, \Pi_\alpha, [\Pi_N]^i, \Pi^{ij}\}$, they are constraints on the initial conditions.

To understand these equations, we investigate the functional derivatives of the matter Hamiltonian with respect to the lapse and shift. We haven't written down the matter Hamiltonian, but we do recall the rule from Hamiltonian mechanics that the partial derivative of a Hamiltonian is given by $\partial H/\partial q|_p = -\partial L/\partial q|_{\dot q}$. Therefore,

$$\left.\frac{\delta H_{\rm matter}}{\delta g_{\mu\nu}}\right|_{\Pi_\alpha,[\Pi_N]^i,\Pi^{ij}} = -\left.\frac{\delta L_{\rm matter}}{\delta g_{\mu\nu}}\right|_{\dot g} = -\frac{1}{2}T^{\mu\nu}\sqrt{-g}. \tag{27}$$

Recalling the form of the metric tensor, we see that since $g_{00} = \gamma^{ij}N_iN_j - \alpha^2$ and $g_{0i} = N_i$:

$$\frac{\delta H_{\rm matter}}{\delta\alpha} = -2\alpha\frac{\delta H_{\rm matter}}{g_{00}} = \alpha\sqrt{-g}\,T^{00} = \alpha^2\sqrt{\gamma}\,T(\boldsymbol{dt},\boldsymbol{dt}) = \sqrt{\gamma}\,T(\boldsymbol{n},\boldsymbol{n}). \tag{28}$$

Similarly, we find that

$$\begin{aligned}
\frac{\delta H_{\rm matter}}{\delta N_i} &= 2\frac{\delta H_{\rm matter}}{\delta g_{0i}} + 2\gamma^{ij}N_j\frac{\delta H_{\rm matter}}{\delta g_{00}} \\
&= -\sqrt{-g}\,\left(T^{0i} + N^iT^{00}\right) \\
&= -\alpha\sqrt{\gamma}\,\left(g^{ij}T^0{}_j + g^{i0}T^0{}_0 + N^iT^0{}_0g^{00} + N^iT^0{}_jg^{0j}\right) \\
&= -\alpha\sqrt{\gamma}\,\left[\gamma^{ij}T^0{}_j - \alpha^{-2}N^iN^jT^0{}_j + \alpha^{-2}N^iT^0{}_0 - N^iT^0{}_0\alpha^{-2} + N^iT^0{}_j\alpha^{-2}N^j\right] \\
&= -\alpha\sqrt{\gamma}\,\gamma^{ij}T^0{}_j \\
&= \sqrt{\gamma}\,\gamma^{ij}T(\boldsymbol{n},\boldsymbol{e}_j).
\end{aligned} \tag{29}$$

Thus $\delta H_{\rm matter}/\delta\alpha = \sqrt{\gamma}\,\rho$ and $\delta H_{\rm matter}/\delta N_i = \sqrt{\gamma}\,J^i$, where ρ is the energy density measured by a normal observer and J^i is the (3-vector) momentum density seen by that observer.

We may now write Eqs. (25,26) directly in terms of the extrinsic curvatures: multiplying them by $8\pi\gamma^{-1/2}$, they are

$$\frac{1}{2}\left[{}^{(D-1)}R + K^2 - K^i_j K^j_i\right] = 8\pi\gamma^{-1/2}\frac{\delta H_{\rm matter}}{\delta\alpha} = 8\pi\rho \quad \text{and} \quad K^{|i} - K^{ij}{}_{|j} = 8\pi\gamma^{-1/2}\frac{\delta H_{\rm matter}}{\delta N_i} = 8\pi J^i. \tag{30}$$

The left hand sides of these equations are $\mathbf{G}(\boldsymbol{n},\boldsymbol{n})$ and $\gamma^{ik}\mathbf{G}(\boldsymbol{n},\boldsymbol{e}_k)$, respectively, according to the relations derived in the previous lecture.

We thus conclude that **the secondary constraints are 4 of the 10 components of Einstein's equations**.

We may further conclude by inspection that the Hamiltonian, at least of GR (this turns out to be true including gauge-invariant matter fields as well), is linear in α and N_i; thus

$$H = \int\left(\alpha\frac{\delta H}{\delta\alpha} + N_i\frac{\delta H}{\delta N_i}\right)d^3x. \tag{31}$$

Thus the Hamiltonian vanishes if the constraints are satisfied. **The actual value of the Hamiltonian for any legal configuration of the Universe is zero.**

One may then wonder if there are additional constraints beyond the secondary constraints. The answer turns out to be no: if the secondary constraints are satisfied at the initial time, then it turns out that their derivatives are zero by the equations of motion, and the constraints are satisfied at all times. (This is exactly true analytically – but may not be true in a numerical code due to round-off errors or numerical integrators with finite step size. In particular, the avoidance of growing constraint violations is a major challenge for numerical codes.)

IV. DYNAMICS

The Hamiltonian, Eq. (23), is most conveniently written as

$$H_{\rm GR} = \int\alpha\left[16\pi G_{ijkl}\Pi^{ij}\Pi^{kl}\gamma^{-1/2} - \frac{{}^{(D-1)}R}{16\pi}\sqrt{\gamma}\right]d^3x + 2\int\Pi^{ij}N_{i|j}\,d^3x, \tag{32}$$

where

$$G_{ijkl} \equiv \gamma_{i(k}\gamma_{l)j} - \frac{1}{D-2}\gamma_{ij}\gamma_{kl} \tag{33}$$

is a tensor symmetric under $i\leftrightarrow j$, $k\leftrightarrow l$, and $ij\leftrightarrow kl$.

From this, we may determine the time evolution of the spatial metric. Since under our assumptions the matter Hamiltonian does not depend explicitly on Π^{ij} (or generally the derivatives of the metric):

$$\dot\gamma_{ij} = \frac{\delta H_{\rm GR}}{\delta\Pi^{ij}} = 32\pi\alpha G_{ijkl}\Pi^{kl}\sqrt{\gamma} + 2N_{(i|j)} \tag{34}$$

or

$$\dot\gamma_{ij} = \frac{\delta H_{\rm GR}}{\delta\Pi^{ij}} = 32\pi\alpha\left(\Pi_{ij} - \frac{1}{D-2}\Pi\gamma_{ij}\right)\sqrt{\gamma} + N_{i|j} + N_{j|i}. \tag{35}$$

This is the usual relation between the derivative of the spatial metric and the extrinsic curvature.

The evolution of the conjugate momentum is trickier. In general it is $\dot\Pi^{ij} = -\delta H/\delta\gamma_{ij}$. There is a part associated with the matter Hamiltonian. Since $g_{00} = \gamma^{kl}N_kN_l - \alpha^2$, we see that

$$-\frac{\delta H_{\rm matter}}{\delta\gamma_{ij}} = -\frac{\delta H_{\rm matter}}{\delta g_{ij}} + \frac{\delta H_{\rm matter}}{\delta g_{00}}\gamma^{ki}\gamma^{jl}N_kN_l = \frac{1}{2}\left(T^{ij} - T^{00}N^iN^j\right)\alpha\sqrt{\gamma}. \tag{36}$$

Now we define the 3-dimensional stress tensor to be the tensor $\mathbf{S}$ on Σ_t to be the restriction of the 4-dimensional stress-energy tensor: $S_{kl} = T_{kl}$. This object is invariant under changes of slices other than Σ_t, and its indices are raised and lowered according to γ_{ij} (note: $S^{ij} \neq T^{ij}$). Then

$$S_{ij} = T_{ij} = g_{i\mu}g_{j\nu}T^{\mu\nu} = N_iN_jT^{00} + 2N_{(i}\gamma_{j)k}T^{0k} + \gamma_{ik}\gamma_{jl}T^{kl}. \tag{37}$$

Raising the indices on both sides with γ^{ij} gives

$$S^{ij} = T^{00}N^iN^j + 2N^{(i}T^{j)0} + T^{ij}. \tag{38}$$

But from Eq. (29) we found that $J^i = \alpha(T^{0i} + N^iT^{00})$. Thus

$$N^{(i}J^{j)} = \alpha(N^{(i}T^{j)0} + N^iN^jT^{00}). \tag{39}$$

We thus conclude that

$$S^{ij} = T^{00}N^iN^j + 2(\alpha^{-1}N^{(i}J^{j)} - N^iN^jT^{00}) + T^{ij} = T^{ij} - T^{00}N^iN^j + 2\alpha^{-1}N^{(i}J^{j)}, \tag{40}$$

and so Eq. (36) gives

$$-\frac{\delta H_{\rm matter}}{\delta\gamma_{ij}} = \frac{1}{2}\left(\alpha S^{ij} - N^iJ^j - N^jJ^i\right)\sqrt{\gamma}. \tag{41}$$

We also need the derivatives of the GR Hamiltonian. Recalling that

$$\frac{\delta}{\delta\gamma_{ij}}\int {}^{(D-1)}R\sqrt{\gamma}\,d^3x = -{}^{(D-1)}G^{ij}\sqrt{\gamma} \tag{42}$$

and keeping track of the numbers of derivatives, we can see that in general

$$\frac{\delta}{\delta\gamma_{ij}}\int \alpha^{(D-1)}R\sqrt{\gamma}\,d^3x = -\alpha^{(D-1)}G^{ij}\sqrt{\gamma} + (c_1\alpha^{|ij} + c_2\gamma^{ij}\gamma^{kl}\alpha_{|kl})\sqrt{\gamma} \tag{43}$$

for some c_1, c_2. One can find c_1 and c_2 by noting that for nearly Euclidean spaces in Cartesian coordinates with $\gamma_{ij} = \delta_{ij} + h_{ij}$ we have ${}^{(D-1)}R = h_{ik,jk} - h_{ij,kk}$; integration by parts in Eq. (43) to move the derivatives onto α gives $c_1 = 1$ and $c_2 = -1$.

Remembering that $\partial\gamma^{-1/2}/\partial\gamma_{ij} = -\frac{1}{2}\gamma^{ij}\gamma^{-1/2}$, we see that

$$\frac{\delta}{\delta\gamma_{ij}}\int \alpha G_{klmn}\Pi^{kl}\Pi^{mn}\,\gamma^{-1/2}\,d^3x = \alpha\left[2\Pi^i_k\Pi^{kj} - \frac{2}{D-2}\Pi\Pi^{ij} - \frac{1}{2}\Pi^k_l\Pi^l_k\gamma^{ij} - \frac{1}{2(D-2)}\Pi^2\gamma^{ij}\right]\gamma^{-1/2}. \tag{44}$$

Finally, the integral involving the shift depends on γ_{ij} only indirectly through the Christoffel symbol on the covariant derivative:

$$\begin{aligned}
2\frac{\delta}{\delta\gamma_{ij}}\int \Pi^{kl}N_{k|l}\,d^3x &= -2\frac{\delta}{\delta\gamma_{ij}}\int \Pi^{kl}N_m{}^{(D-1)}\Gamma^m_{kl}\,d^3x \\
&= -\frac{\delta}{\delta\gamma_{ij}}\int \Pi^{kl}N_m\gamma^{mn}(-\gamma_{kl,n} + \gamma_{nk,l} + \gamma_{nl,k})\,d^3x \\
&= \Pi^{kl}N_m\gamma^{m(i}\gamma^{j)n}(-\gamma_{kl,n} + \gamma_{nk,l} + \gamma_{nl,k}) - (\Pi^{ij}N_m\gamma^{mn})_{,n} + (\Pi^{li}N_m\gamma^{jm})_{,l} + (\Pi^{lj}N_m\gamma^{im})_{,l} \\
&= 2{}^{(D-1)}\Gamma^{(j}_{kl}N^{i)}\Pi^{kl} - (\Pi^{ij}N^n)_{,n} + 2(\Pi^{l(i}N^{j)})_{,l} \\
&= 2{}^{(D-1)}\Gamma^{(j}_{kl}N^{i)}\Pi^{kl} - (\Pi^{ij}\gamma^{-1/2}N^n)_{,n}\gamma^{1/2} + 2(\Pi^{l(i}N^{j)}\gamma^{-1/2})_{,l}\gamma^{1/2} \\
&\quad -{}^{(D-1)}\Gamma^k_{kn}\Pi^{ij}N^n + 2{}^{(D-1)}\Gamma^k_{kl}\Pi^{l(i}N^{j)}. \\
&= -\gamma^{1/2}(N^l\Pi^{ij}\gamma^{-1/2})_{|l} + 2(N^{(j}\Pi^{i)l}\gamma^{-1/2})_{|l}\gamma^{1/2}.
\end{aligned} \tag{45}$$

Putting all the pieces together, we find that

$$\begin{aligned}
\dot\Pi^{ij} &= -16\pi\alpha\left[2\Pi^i_k\Pi^{kj} - \frac{2}{D-2}\Pi\Pi^{ij} - \frac{1}{2}\Pi^k_l\Pi^l_k\gamma^{ij} - \frac{1}{2(D-2)}\Pi^2\gamma^{ij}\right]\gamma^{-1/2} - \frac{\alpha}{16\pi}{}^{(D-1)}G^{ij}\sqrt{\gamma} \\
&\quad + \frac{1}{16\pi}(\alpha^{|ij} - \gamma^{ij}\gamma^{kl}\alpha_{|kl})\sqrt{\gamma} + \gamma^{1/2}(N^l\Pi^{ij}\gamma^{-1/2})_{|l} - (N^j\Pi^{il}\gamma^{-1/2})_{|l}\gamma^{1/2} - (N^i\Pi^{jl}\gamma^{-1/2})_{|l}\gamma^{1/2} \\
&\quad + \frac{1}{2}\left(\alpha S^{ij} - N^iJ^j - N^jJ^i\right)\sqrt{\gamma}.
\end{aligned} \tag{46}$$

Note that aside from the final factor of $\sqrt{\gamma}$, this is a tensor on Σ_t.

We thus see how the spatial metric and its conjugate momentum (and hence the extrinsic curvature) evolve. We also know that they must satisfy the initial value (secondary) constraints, Eq. (30). This leaves open the issue of the evolution of α and N_i, whose conjugate momenta are zero. Of course, we want to write down a rule like

$$\dot{\alpha} = \frac{\delta H}{\delta \Pi_\alpha}, \tag{47}$$

but since Π_α is identically zero, the functional derivative is undefined. Mathematically, the functional derivative can be anything since we have not defined H for $\Pi_\alpha \neq 0$, and so we conclude that $\dot{\alpha}$ and $\dot{N}_i$ can be anything. This is nothing but the gauge ambiguity of GR: the spatial slice Σ_t can be pushed forward and re-parameterized according to any α and N_i functions that we choose.

More generally, this phenomenon is part of the behavior of Legendre transformations of an action principle when many paths related by a gauge transformation have the same action. Remember that the conjugate momentum is the derivative of the action with respect to final coordinates. The resulting primary constraint that some of the conjugate momenta are always zero (due to gauge-equivalent configurations) allows us to choose when evolving the system which of the many gauge-equivalent descriptions of the final state is explored by our numerical code.