

Lecture XXXIV: Hypersurfaces and the 3+1 formulation of geometry

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I. OVERVIEW

Having covered the Lagrangian formulation of GR, our next goal is to build the conjugate momenta to $g_{\mu\nu}$ and construct a Hamiltonian. Before we do, however, it is useful to consider the subject of slicing a 4-dimensional spacetime into 3-dimensional slices of constant “time.” There are of course many such slices, and in numerical computations we will ultimately have to choose one.

So our task for this lecture is to study the subject of surfaces embedded in spaces of one higher dimension.

Reading:

- MTW §§21.4–21.5.

II. SPLITTING THE METRIC TENSOR

We begin with the study of the following notations in D -dimensional spacetime (we care most about $D = 4$, but we will occasionally make analogies to 2-surfaces in 3-dimensional space, so it is useful to keep D general). There is an overall D -dimensional metric tensor $g_{\mu\nu}$. If we choose a slicing of the spacetime into surfaces Σ_t , then clearly there is a $D - 1$ -dimensional metric on each surface whose components are simply the $(D - 1) \times (D - 1)$ submatrix of g . This metric tensor will be denoted by γ_{ij} . Thus $g_{\mu\nu}$ is partitioned:

$$g_{\mu\nu} = \left(\begin{array}{c|c} A & N_i \\ \hline N_i & \gamma_{ij} \end{array} \right), \quad (1)$$

where $A = g_{tt}$ and $N_i = g_{it}$. There is a set of coordinate basis vectors, $\mathbf{e}_t$ and $\{\mathbf{e}_i\}_{i=1}^{D-1}$; the $\mathbf{e}_i$ are tangent to Σ_t , while $\mathbf{e}_t$ is not tangent to Σ_t .

A. First fundamental form

The $D - 1$ -dimensional metric γ_{ij} on Σ_t is called the *first fundamental form*. It can be viewed as a symmetric, linear operation at each point $\mathcal{P} \in \Sigma_t$ – that is, $\gamma : T_{\mathcal{P}}\Sigma_t \times T_{\mathcal{P}}\Sigma_t \rightarrow \mathbb{R}$. (The name will distinguish it from the second fundamental form, which we define later.)

It is conventional to raise and lower the indices of vectors, tensors, etc. defined on Σ_t with respect to the spatial metric, i.e. the first fundamental form. This is valid so long as the surface is not null, i.e. so that γ_{ij} is nonsingular. Thus, γ^{ij} is the $(D - 1) \times (D - 1)$ matrix inverse of γ_{ij} . It does **not** need to be equal to g^{ij} , which is the inverse of the $D \times D$ metric.

B. Shift

The specification of γ_{ij} provides the $(D - 1)$ -dimensional geometry on each hypersurface, but A and N_i are required in order to describe how these hypersurfaces are connected to each other. Note that

$$N_i = \mathbf{e}_t \cdot \mathbf{e}_i; \quad (2)$$

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this is therefore a measure of how much the trajectory of a particle at constant spatial coordinates is non-orthogonal to the surface Σ_t . If the curves of constant x^i (which we term *threads*) are orthogonal to the slices of constant t , then $N_i = 0$. We therefore call N_i the *shift*. The shift is actually a 1-form (linear mapping : $T\mathcal{P}\Sigma_t \rightarrow \mathbb{R}$) on the space of vectors tangent to Σ_t : it is defined by

$$\tilde{N}(\mathbf{v}) = \mathbf{e}_t \cdot \mathbf{v} \quad (3)$$

for $\mathbf{v} \in T\mathcal{P}\Sigma_t$.

One may introduce the contravariant shift, which is a vector tangent to Σ_t ,

$$N^i \equiv \gamma^{ij} N_j. \quad (4)$$

C. Lapse

Next consider the forward-directed unit normal to the surfaces, $\mathbf{n}$. This has square norm

$$s \equiv \mathbf{n} \cdot \mathbf{n}; \quad (5)$$

of course, for spacelike surfaces in Minkowski-signature spacetime as we are considering, $s = -1$, but if we were to consider surfaces in Euclidean-signature space (which we will do as examples) we have $s = +1$. Since $\mathbf{n}$ is normal to the surface, it follows that for $\mathbf{v} \in T\mathcal{P}\Sigma_t$ (i.e. $\mathbf{v}^t = 0$) we have $\mathbf{n} \cdot \mathbf{v} = 0$; therefore $n_i = 0 \ \forall i \neq t$. We may thus write

$$n_t = s\alpha, \quad (6)$$

where s was included to meet the forward-directed condition and

$$\alpha = -\mathbf{e}_t \cdot \mathbf{n} \quad (7)$$

is called the *lapse*. (MTW denotes the lapse by N , which is common but dislikable since it can be confused with the shift.)

We can understand the lapse better if we make some additional observations. First, we see that

$$s\alpha^{-2} = \alpha^{-2} \mathbf{n} \cdot \mathbf{n} = s \, d\mathbf{t} \cdot s \, d\mathbf{t} = d\mathbf{t} \cdot d\mathbf{t} = g^{tt}, \quad (8)$$

so the contravariant metric component g^{tt} is related to the lapse. We may use this to raise the index on $\mathbf{n}$:

$$n^t = \alpha^{-1}. \quad (9)$$

It follows that an observer moving orthogonal to the $\{\Sigma_t\}$ – i.e. with 4-velocity $\mathbf{n}$ – sees a relation between coordinate time and proper time of $dt/d\tau = \alpha^{-1}$ or $d\tau/dt = \alpha$. The lapse is thus interpreted as the proper time per unit coordinate time seen by an observer moving orthogonal to the slices. (In the Euclidean-signature case, we should say this is the proper distance per unit coordinate distance – i.e. the proper spacing between slices divided by the coordinate change Δt .)

We still have not specified how the lapse relates to A , N_i , and γ_{ij} . For this we need to construct the inverse D -metric.

D. Inverse metric and line element

We will make frequent use of the inverse metric, which may be obtained by partitioning formula:

$$\left(\begin{array}{c|c} \mathbf{P} & \mathbf{Q} \\ \hline \mathbf{R} & \mathbf{S} \end{array} \right)^{-1} = \left(\begin{array}{c|c} (\mathbf{P} - \mathbf{Q}\mathbf{S}^{-1}\mathbf{R})^{-1} & -(\mathbf{P} - \mathbf{Q}\mathbf{S}^{-1}\mathbf{R})^{-1}\mathbf{Q}\mathbf{S}^{-1} \\ \hline -\mathbf{S}^{-1}\mathbf{R}(\mathbf{P} - \mathbf{Q}\mathbf{S}^{-1}\mathbf{R})^{-1} & \mathbf{S}^{-1} + \mathbf{S}^{-1}\mathbf{R}(\mathbf{P} - \mathbf{Q}\mathbf{S}^{-1}\mathbf{R})^{-1}\mathbf{Q}\mathbf{S}^{-1} \end{array} \right). \quad (10)$$

(Try explicit multiplication.) Applying this to the partition of the covariant metric, Eq. (1), yields first the relation

$$s\alpha^{-2} = g^{tt} = (A - N_i \gamma^{ij} N_j)^{-1} = (A - N_i N^i)^{-1}, \quad (11)$$

or solving for A :

$$g_{tt} = A = N_i N^i + s\alpha^2. \quad (12)$$

This establishes the relation between the lapse and the time-time component of the metric. Note again that N^i is raised using the $(D-1)$ -dimensional metric γ^{ij} .

The lower-left entry in Eq. (10) gives, recalling that $g^{tt} = (\mathbf{P} - \mathbf{Q}\mathbf{S}^{-1}\mathbf{R})^{-1}$,

$$g^{it} = -\gamma^{ij}g_{jt}g^{tt} = -\gamma^{ij}N_js\alpha^{-2} = -s\alpha^{-2}N^j. \quad (13)$$

Finally, the partition formula for the space-space components is

$$g^{ij} = \gamma^{ij} + \gamma^{ik}N_ks\alpha^{-2}N_l\gamma^{lj} = \gamma^{ij} + s\alpha^{-2}N^iN^j. \quad (14)$$

In conclusion, the metric and its inverse decompose as

$$g_{\mu\nu} = \left(\begin{array}{c|c} N_iN^i + s\alpha^2 & N_i \\ \hline N_i & \gamma_{ij} \end{array} \right) \quad (15)$$

and

$$g^{\mu\nu} = \left(\begin{array}{c|c} s\alpha^{-2} & -s\alpha^{-2}N^i \\ \hline -s\alpha^{-2}N^i & \gamma^{ij} + s\alpha^{-2}N^iN^j \end{array} \right). \quad (16)$$

The line element form is

$$ds^2 = s\alpha^2 dt^2 + \gamma_{ij}(dx^i + N^i dt)(dx^j + N^j dt). \quad (17)$$

E. Commonly used vectors

In our investigation of “initial conditions,” the normal vector will play a more important role than $\mathbf{e}_t$. This is because once Σ_t is specified at a particular $t = t_{\text{init}}$, $\mathbf{n}$ as a geometric object is uniquely determined, whereas $\mathbf{e}_t$ is not – thus we can describe initial conditions (e.g. the electromagnetic potential) in a basis that contains $\mathbf{n}$ rather than $\mathbf{e}_t$. Obtaining the relevant conversions between the two choices is thus critical.

Let us first consider $\mathbf{n}$. We know its lower-index version $n_\mu = (s\alpha, \mathbf{0})$. The raised index version is then $n^\mu = (\alpha^{-1}, -\alpha^{-1}N^i)$, and we thus see that

$$\mathbf{n} = \frac{1}{\alpha}(\mathbf{e}_t - N^i\mathbf{e}_i). \quad (18)$$

Of equal interest is the solution for $\mathbf{e}_t$ in terms of $\mathbf{n}$ and $\mathbf{e}_i$:

$$\mathbf{e}_t = \alpha\mathbf{n} + N^i\mathbf{e}_i. \quad (19)$$

This is the key equation providing a simple physical interpretation of the lapse and shift: they describe the evolution of the coordinate system. To get from a point (t, x^i) to $(t + \Delta t, x^i)$, one moves a distance $\alpha\Delta t$ orthogonal to the hypersurface Σ_t , and shifts parallel to the surface by a vector displacement $N^i\mathbf{e}_i\Delta t$. If one goes to the point $(t + \Delta t, x^i)$ and then drops a perpendicular to the surface Σ_t , the nearest point is not (t, x^i) but $(t, x^i + N^i\Delta t)$. Hence the name “shift.”

There is an alternative statement of what the shift means, in the Minkowskian signature spacetime case ($s = -1$). Consider two observers, one who moves along a thread (i.e. at fixed x^i – the *coordinate observer*) and one who moves normal to the hypersurfaces (the *normal observer*). Their 4-velocities are respectively $(-g_{tt})^{-1/2}\mathbf{e}_t$ and $\mathbf{n}$. We may find their relative velocity V by taking the dot product of the 4-velocities and setting it equal to the Lorentz factor $(1 - V^2)^{-1/2}$:

$$(1 - V^2)^{-1/2} = -(-g_{tt})^{-1/2}\mathbf{e}_t \cdot \mathbf{n} = -(-g_{tt})^{-1/2}n_t = -(\alpha^2 - N_iN^i)^{-1/2}(-\alpha) = \left(1 - \frac{N_iN^i}{\alpha^2}\right)^{-1/2}, \quad (20)$$

or

$$V = \frac{\sqrt{N_iN^i}}{\alpha}. \quad (21)$$

Thus the relative velocity of these two observers is the ratio of the magnitude of the $(D-1)$ -dimensional shift vector to the lapse. If the magnitude of the shift exceeds the lapse, then the coordinate grid (the labels x^1, x^2, x^3) moves faster than the speed of light, and the coordinate observer is no longer a legal observer. This is perfectly fine: the coordinate grid is our own mathematical invention, and it can move however we choose!

Since the lapse and shift merely describe the evolution of the coordinate system, you might guess – correctly – that in the time evolution formulations of GR, we are free to choose them for convenience.

III. EXTRINSIC CURVATURE

Thus far we have discussed the intrinsic geometry of a 3-dimensional surface (γ_{ij}) and the methods by which many such surfaces are stitched together to make 4-dimensional spacetime. It is clear that this process, by defining all components of the metric $g_{\mu\nu}(x^\alpha)$, completely specifies the spacetime geometry. However, when we specify initial conditions, it will be on a particular surface Σ_t . How do we specify how Σ_t is embedded in the spacetime without explicitly describing the neighboring slices and their spatial metrics? Or, if we consider a piece of paper in 3-dimensional Euclidean space, how do we specify that the paper is bent into a cylinder instead of being truly flat? This brings us to the subject of “extrinsic curvature,” a property of any $D - 1$ -surface Σ in D -dimensional spacetime $\mathcal{M}$ that exists without reference to other surfaces. The specification of the extrinsic curvature as initial data will take the place of the specification of $\dot{\gamma}_{ij}$ (which depends on the lapse and shift as well as on the choice of Σ_t).

The concept is most easily explained by considering a 2-dimensional surface in Euclidean 3-space $\mathbb{R}^3$. Take a surface tangent to the xy -plane at the origin; if it is smooth it may be Taylor-expanded as

$$z = \frac{1}{2}K_{xx}x^2 + K_{xy}xy + \frac{1}{2}K_{yy}y^2 + \dots; \quad (22)$$

the components of the 2×2 symmetric matrix $\mathbf{K}$ describe how the surface is curved with respect to the manifold in which it is embedded. It is this concept that we wish to generalize to any surface. We see that if we move in e.g. the x -direction, then the surface normal tilts in the x or y directions depending on K_{xx} and K_{xy} respectively. This forms the basis of the general definition.

A. Definition and immediate implications

We define the *extrinsic curvature* of a $D - 1$ -surface $\Sigma_t \subset \mathcal{M}$ at a point $\mathcal{P} \in \Sigma_t$ as the mapping $\mathbf{K} : T_{\mathcal{P}}\Sigma_t \rightarrow T_{\mathcal{P}}\mathcal{M}$,

$$\mathbf{K}(\mathbf{v}) = -\nabla_{\mathbf{v}}\mathbf{n}. \quad (23)$$

Here $\nabla_{\mathbf{v}}$ is the usual D -dimensional covariant derivative. Note that since $\mathbf{v} \in T_{\mathcal{P}}\Sigma_t$, we do not care what the value of $\mathbf{n}$ anywhere except on Σ_t : thus this definition depends only on Σ_t and not on the lapse and shift. Moreover, we see that

$$\mathbf{n} \cdot \mathbf{K}(\mathbf{v}) = -\mathbf{n} \cdot \nabla_{\mathbf{v}}\mathbf{n} = -\frac{1}{2}\nabla_{\mathbf{v}}(\mathbf{n} \cdot \mathbf{n}) = 0. \quad (24)$$

Therefore, $\mathbf{K}(\mathbf{v})$ is tangent to Σ_t and we learn that $\mathbf{K}$ is actually a mapping from $T_{\mathcal{P}}\Sigma_t$ to itself. It is thus a $(D - 1) \times (D - 1)$ matrix, with components K^i_j . It is called the *extrinsic curvature* or *second fundamental form*.

Note that $\mathbf{K}$ flips sign if we flip the direction of $\mathbf{n}$.

At first glance, it would appear that all of the entries of $\mathbf{K}$ are independent. This is not so: if we take two vectors $\mathbf{u}, \mathbf{v} \in T_{\mathcal{P}}\Sigma_t$, we see that

$$\begin{aligned} \mathbf{u} \cdot \mathbf{K}(\mathbf{v}) - \mathbf{v} \cdot \mathbf{K}(\mathbf{u}) &= -u^\alpha v^\beta n_{\alpha;\beta} + v^\alpha u^\beta n_{\alpha;\beta} \\ &= 2v^\alpha u^\beta n_{[\alpha;\beta]} \\ &= 2v^\alpha u^\beta n_{[\alpha,\beta]} = 0 \end{aligned} \quad (25)$$

since $v^t = u^t = 0$ and $n_i = 0$. We thus see that $\mathbf{u} \cdot \mathbf{K}(\mathbf{v}) = u^i \gamma_{ik} K^k_j v^j$ is symmetric in $\mathbf{u}$ and $\mathbf{v}$, or equivalently K_{ij} (indices lowered by γ_{ij}) is symmetric. Thus $\mathbf{K}$ is a symmetric form.

Note that while the first fundamental form γ_{ij} was a property of only Σ_t without reference to $\mathcal{M}$, the second fundamental form uses the embedding in an essential way. Thus a 3-sphere as its own geometric entity has a first fundamental form but not a second, while if it is embedded in 4-dimensional spacetime (e.g. a closed FRW universe), it has both.

A corollary to Eq. (23) is that the extrinsic curvature describes the apparent relative velocities of neighboring normal observers. In particular, if $\mathcal{O}$ and $\mathcal{P}$ are infinitesimally displaced from each other by the vector $\boldsymbol{\xi}$ pointing from $\mathcal{O}$ to $\mathcal{P}$ (this construction only makes sense for infinitesimal separations) then the apparent velocity of the normal observer at $\mathcal{P}$ relative to that at $\mathcal{O}$ is $-\mathbf{K}(\boldsymbol{\xi})$. In the special case of an FRW universe, where this velocity is $H\boldsymbol{\xi}$, we must have an extrinsic curvature of $K_{ij} = -H\gamma_{ij}$. Thus the extrinsic curvature provides a measure of the rate of contraction of observers moving normal to the surface, with its trace giving a measure of “ $-3H$ ” (the Hubble rate) and its anisotropic part measuring anisotropic expansion/contraction. There is no antisymmetric or rotational part: the requirement of being hypersurface-orthogonal forbids our observers from having any vorticity.

B. Relation to the metric

So much for defining the extrinsic curvature: we must now determine how it relates to the “time rate of change of the metric” $\dot{\gamma}_{ij}$ (which we expect to appear in any time evolution version of Einstein’s equations).

All we have to do is evaluate Eq. (23). This gives

$$K_{ij} = -\nabla_j n_i = -n_{i,j} + \Gamma_{ij}^k n_k, \quad (26)$$

where we use Γ to denote D -dimensional Christoffel symbols. Since only n_t is nonvanishing and has $n_t = s\alpha$, we find

$$\begin{aligned} K_{ij} &= s\alpha\Gamma_{ij}^t \\ &= \frac{1}{2}s\alpha g^{t\mu}(-g_{ij,\mu} + g_{i\mu,j} + g_{j\mu,i}) \\ &= \frac{1}{2}s\alpha [(s\alpha^{-2})(-\dot{\gamma}_{ij} + N_{i,j} + N_{j,i}) + (-s\alpha^{-2}N^k)(-\gamma_{ij,k} + \gamma_{ik,j} + \gamma_{jk,i})]. \end{aligned} \quad (27)$$

The solution for $\dot{\gamma}_{ij}$ is

$$\dot{\gamma}_{ij} = -2\alpha K_{ij} + N_{i,j} + N_{j,i} + N^k(-\gamma_{ij,k} + \gamma_{ik,j} + \gamma_{jk,i}). \quad (28)$$

Noting that $N^k = \gamma^{kl}N_l$, we can collect the last term into a $D-1$ -dimensional Christoffel symbol: it is $2^{(D-1)}\Gamma_{ij}^l N_l$. If we therefore introduce the use of the vertical bar $|$ to denote covariant derivatives on Σ_t using the spatial metric, we reduce this to

$$\dot{\gamma}_{ij} = -2\alpha K_{ij} + N_{i|j} + N_{j|i}. \quad (29)$$

This is the time evolution equation for γ_{ij} – provided that we can determine K_{ij} . To close the time evolution formalism, we will need to find $\dot{K}_{ij}$. This is most easily done through Hamiltonian means, since it will turn out that the components of K^{ij} are closely related to the canonical conjugates of γ_{ij} .

At a more general level, Eq. (29) gives us another concept of what extrinsic curvature is. Suppose we take a surface with some metric γ_{ij} , and then make a “sandwich” between it and another surface a distance ϵ away from it (i.e. take the unit normal at each point and travel a distance ϵ in that direction; then use the new points to define a new surface). We see that the new surface has metric $\gamma_{ij} - 2\epsilon K_{ij}$.

IV. GAUSS-CODAZZI RELATIONS

If you have ever bent a piece of paper initially in the xy -plane, you know that it is easy to bend it in the x -direction, or the y -direction, but not both simultaneously. This trivial geometric realization is embodied in a set of beautiful relations between the various curvature tensors we have encountered: those associated with the $D-1$ -dimensional surface Σ_t , those associated with the overall spacetime, and the extrinsic curvature. These relations also play a central role in defining legal initial conditions in GR. We now examine them. We denote the Riemann tensor on the hypersurface by ${}^{(D-1)}R_{ijkl}$, as opposed to those in the spacetime, which are $R_{\alpha\beta\gamma\delta}$.

A. Derivation

Our approach will be to take the surface Σ_t and build *Gaussian normal coordinates* – a coordinate system formed by taking $\alpha = 1$ and $N_i = 0$ everywhere. [Such a system always exists: one merely takes each spatial position x^i on Σ_t , builds the geodesic orthogonal to Σ_t , and takes these as the threads; and chooses the proper time as the coordinate – see MTW Exercise 27.2.] At the end of the calculation we will write our expressions such that they are coordinate-independent, and thus valid in any system.

In the Gaussian normal system, we see that the Christoffel symbols take on a special form. We immediately see that

$$\Gamma_{ij}^t = sK_{ij} \quad \text{and} \quad \Gamma_{jk}^i = {}^{(D-1)}\Gamma_{jk}^i; \quad (30)$$

see Eq. (27) for the first relation, and in the second relation t plays no role. Furthermore,

$$\Gamma_{tj}^i = \frac{1}{2}\gamma^{ik}\dot{\gamma}_{kj} = -\gamma^{ik}K_{kj} = K_j^i. \quad (31)$$

All Christoffel symbols with 2 or 3 t s vanish.

Now using the general rule for the Riemann tensor,

$$R^\alpha{}_{\beta\gamma\delta} = \Gamma^\alpha_{\beta\delta,\gamma} - \Gamma^\alpha_{\beta\gamma,\delta} + \Gamma^\alpha_{\mu\gamma}\Gamma^\mu_{\beta\delta} - \Gamma^\alpha_{\mu\delta}\Gamma^\mu_{\beta\gamma}, \quad (32)$$

we may consider the components of the D -dimensional Riemann tensor. For the purely spatial terms, this gives

$$\begin{aligned} R^i{}_{jkl} &= {}^{(D-1)}\Gamma^i{}_{jl,k} - {}^{(D-1)}\Gamma^i{}_{jk,l} + {}^{(D-1)}\Gamma^i{}_{mk} {}^{(D-1)}\Gamma^m{}_{jl} - sK^i_k K_{jl} - {}^{(D-1)}\Gamma^i{}_{ml} {}^{(D-1)}\Gamma^m{}_{jk} + sK^i_l K_{jk} \\ &= {}^{(D-1)}R^i{}_{jkl} + s(K^i_l K_{jk} - K^i_k K_{jl}). \end{aligned} \quad (33)$$

Lowering an index (which, in Gaussian normal coordinates, is the same according to either γ_{ij} or g_{ij}) gives the relation

$$R_{ijkl} = {}^{(D-1)}R_{ijkl} + s(K_{il}K_{jk} - K_{ik}K_{jl}). \quad (34)$$

Since $R_{ijkl} = R(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k, \mathbf{e}_l)$ does not depend on the choice of $\mathbf{e}_0$ (and hence the lapse and shift), Eq. (34) applies to a general coordinate system. It is the *first Gauss-Codazzi relation*.

A second such relation can be obtained by considering the t component in Gaussian normal coordinates:

$$\begin{aligned} R^t{}_{ijk} &= sK_{ik,j} - sK_{ij,k} + sK_{lj} {}^{(D-1)}\Gamma^l{}_{ik} - sK_{lk} {}^{(D-1)}\Gamma^l{}_{ij} \\ &= s(K_{ik,j} - K_{ij,k} + K_{lj} {}^{(D-1)}\Gamma^l{}_{ik} - K_{lk} {}^{(D-1)}\Gamma^l{}_{ij}) \\ &= s(K_{ik|j} - K_{ij|k}). \end{aligned} \quad (35)$$

In Gaussian normal coordinates, $n_\mu = (s, \mathbf{0})$ so this is equal to $sn_\mu R^\mu{}_{ijk}$. We may thus write the following relation, which is coordinate-independent:

$$n_\mu R^\mu{}_{ijk} = K_{ik|j} - K_{ij|k}. \quad (36)$$

This is the *second Gauss-Codazzi relation*.

B. Relation to the Einstein tensor

Four of the ten components of the Einstein tensor can be extracted from the Gauss-Codazzi relations. This can be shown directly, but it is easiest to specialize to Gaussian normal coordinates and then to a system where the spatial coordinates are re-labeled so as to be locally Euclidean at some point $\mathcal{P}$, prove the appropriate relations, and then transform back to general coordinates. In the intermediate case, we have not only $\alpha = 0$ and $N_i = 0$ but also $\gamma_{ij} = \delta_{ij}$ at $\mathcal{P}$ itself.

First up is the time-time component. We see that

$$\begin{aligned} G_{tt} &= R_{tt} - \frac{1}{2}sR \\ &= R_{tt} - \frac{1}{2}s(sR_{tt} + R_{ii}) \\ &= \frac{1}{2}R_{tt} - \frac{1}{2}sR_{ii} \\ &= \frac{1}{2}R_{titi} - \frac{1}{2}sR_{ijjj} - \frac{1}{2}R_{itit} \\ &= -\frac{1}{2}sR_{ijij} \\ &= -\frac{1}{2}s \left[{}^{(D-1)}R_{ijij} + s(K_{ij}K_{ji} - K_{ii}K_{jj}) \right] \\ &= -\frac{1}{2} \left[s {}^{(D-1)}R + K^j_i K^i_j - K^i_i K^j_j \right]. \end{aligned} \quad (37)$$

The left-hand side is the coordinate-independent object $G(\mathbf{n}, \mathbf{n})$, and the right-hand side is manifestly coordinate-independent (so long as we fix Σ_t), so we conclude that

$$G(\mathbf{n}, \mathbf{n}) = -\frac{1}{2} \left[s {}^{(D-1)}R + K^j_i K^i_j - K^2 \right], \quad (38)$$

where $K \equiv K_i^i$ is the $D - 1$ -trace of the extrinsic curvature. Thus the normal component of the Einstein equation, which we will equate to 8π times the energy density seen by a normal observer, is related to the $D - 1$ -dimensional Ricci scalar of the surface and its extrinsic curvature.

Next let's look at the time-space components. These are, in the specialized coordinate system,

$$G_{ti} = R_{ti} = R_{tjij} = sR^t_{jij} = K_{jj|i} - K_{ij|j}. \quad (39)$$

We thus conclude that in general coordinates:

$$G(\mathbf{n}, \mathbf{e}_i) = K_{|i} - K_i^j{}_{|j}. \quad (40)$$

C. Space-space parts of Einstein tensor

The space-space parts of the Einstein tensor do not have any such simple relation. To see this, consider, in Gaussian normal coordinates:

$$\begin{aligned} G_j^i &= R_j^i - \frac{1}{2}R\delta_j^i \\ &= R_j^i - \frac{1}{2}R_k^k\delta_j^i - \frac{1}{2}sR_{tt}\delta_j^i \\ &= R^{ti}{}_{tj} + R^{ki}{}_{kj} - \frac{1}{2}R^{tk}{}_{tk}\delta_j^i - \frac{1}{2}R^{mk}{}_{mk}\delta_j^i - \frac{1}{2}sR^{kt}{}_{kt}\delta_j^i \\ &= R^{ti}{}_{tj} + R^{ki}{}_{kj} - R^{tk}{}_{tk}\delta_j^i - \frac{1}{2}R^{mk}{}_{mk}\delta_j^i \\ &= R^{ti}{}_{tj} + {}^{(D-1)}R^{ki}{}_{kj} + s(K_j^k K_k^i - K_k^k K_j^i) - R^{tk}{}_{tk}\delta_j^i - \frac{1}{2}[{}^{(D-1)}R^{mk}{}_{mk} + s(K_k^m K_m^k - K_m^m K_k^k)]\delta_j^i \\ &= R^{ti}{}_{tj} - R^{tk}{}_{tk}\delta_j^i + {}^{(D-1)}G_j^i + s(K_k^i K_j^k - K K_j^i) - \frac{1}{2}s(K_l^k K_k^l - K^2)\delta_j^i. \end{aligned} \quad (41)$$

The problem is the appearance of the Riemann tensor component with two time indices, $R^{ti}{}_{tj}$. This depends not just on the spatial metric and the extrinsic curvature, but on the geometry of spacetime off of the “sandwich” of a hypersurface and its immediate neighbor. In Gaussian normal coordinates,

$$R^t{}_{itj} = s\dot{K}_{ij} - sK_{kj}K_i^k, \quad (42)$$

so we have the simplified form obtained by lowering indices on Eq. (41),

$$G_{ij} = s(\dot{K}_{ij} - \gamma^{kl}\dot{K}_{kl}\gamma_{ij} - K_{kj}K_i^k + K_{kl}K^{kl}\gamma_{ij}) + {}^{(D-1)}G_{ij} + s(K_{ik}K_j^k - K K_{ij}) - \frac{1}{2}s(K_l^k K_k^l - K^2)\gamma_{ij}, \quad (43)$$

which simplifies to

$$G_{ij} = s(\dot{K}_{ij} - \gamma^{kl}\dot{K}_{kl}\gamma_{ij} + \frac{1}{2}K_{kl}K^{kl}\gamma_{ij} + K K_{ij} - \frac{1}{2}K^2\gamma_{ij}) + {}^{(D-1)}G_{ij}. \quad (44)$$

The appearance of $\dot{K}_{ij}$ is troublesome because it transforms in a complicated way back to a general coordinate system. It has a particular interpretation – it is $-\frac{1}{2}$ times the second derivative of the spatial metric with respect to proper distance if one extends the sandwich construction to a sequence of hypersurfaces – but this is not especially useful. We will therefore not make use of the space-space part of the Einstein tensor in developing the time evolution formulation: rather we will use Hamiltonian methods, which will give us the evolution equations for K_{ij} directly.

D. Some simple applications

We conclude by considering some simple applications of the Gauss-Codazzi relations.

First, we consider the problem of the intrinsic curvature of a 2-dimensional surface in 3-dimensional Euclidean signature space. The 2-dimensional Riemann tensor ${}^{(2)}R_{ijkl}$ is entirely determined by the Ricci scalar ${}^{(2)}R$ and the metric γ_{ij} (the Riemann tensor has only 1 nontrivial component, ${}^{(2)}R_{1212}$). We further suppose that the extrinsic

curvature K_{ij} , being a symmetric 2×2 tensor, has eigenvalues λ_1 and λ_2 (the “principal curvatures”) in an orthonormal basis. Then the Gauss-Codazzi relation says

$$G(\mathbf{n}, \mathbf{n}) = -\frac{1}{2} \left[{}^{(2)}R + \lambda_1^2 + \lambda_2^2 - (\lambda_1 + \lambda_2)^2 \right] = -\frac{1}{2} {}^{(2)}R + \lambda_1 \lambda_2. \quad (45)$$

We thus see that if the 3-dimensional space is flat, then $G(\mathbf{n}, \mathbf{n}) = 0$ and ${}^{(2)}R = 2\lambda_1\lambda_2$. That is, a 2-surface in Euclidean $\mathbb{R}^3$ has positive curvature if the two principal curvatures λ_1 and λ_2 have the same sign (i.e. the surface curves in the same direction on both axes, like a sphere); it has negative curvature if λ_1 and λ_2 have opposite sign (i.e. the surface curves downward on one axis and upward on the other, like a saddle); and zero curvature if one of the principal curvatures is zero (like a cylinder; if both are zero, then the surface has no extrinsic curvature whatsoever). This is the reason why an initially flat piece of paper in Euclidean space can be bent into a cylinder along any axis, but cannot be bent into a ball or a saddle without crumpling or tearing.

The opposite situation occurs if we were to put that flat piece of paper in S^3 , where $G(\mathbf{n}, \mathbf{n}) = -1$ for any unit vector $\mathbf{n}$. In this case, for the piece of paper to be flat, we need $\lambda_1\lambda_2 = -1$, i.e. it would have to have saddle-like extrinsic curvature!

A second example concerns spacetimes with time-reversal symmetry, e.g. a closed FRW universe at the epoch of greatest expansion. If we take the surface Σ of symmetry, then its extrinsic curvature must vanish, and we have

$$8\pi\rho = G(\mathbf{n}, \mathbf{n}) = \frac{1}{2} {}^{(3)}R. \quad (46)$$

Thus, any spacetime with such a time-reversal symmetry must have spatial curvature if it is nonempty, and such curvature is given by ${}^{(3)}R = 16\pi\rho$. If the energy density is non-negative, we must have non-negative ${}^{(3)}R$ at all points on the surface of symmetry.

A third example concerns the subject of *extremal surfaces*, $D - 1$ -dimensional surfaces Σ whose $D - 1$ -volume is stationary with respect to small displacements of the surface (think of the surface of a soap film attached to a boundary). It is straightforward to show that the variation of the surface volume if the surface is perturbed by a distance $w\mathbf{n}$ is

$$\delta {}^{(D-1)}V = - \int wK \sqrt{-\gamma} d^{D-1}x, \quad (47)$$

and hence the extremal surfaces are those for which the trace of the extrinsic curvature vanishes, $K = 0$. If such surfaces are embedded in Euclidean space, then we find the relations

$${}^{(D-1)}R = -K_j^i K_i^j \quad \text{and} \quad K^{ij}|_j = 0. \quad (48)$$

Thus the extrinsic curvature is a divergenceless tensor on any extremal surface on a flat background.