Lecture XXXIII: Lagrangian formulation of GR

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I. OVERVIEW

We now turn our attention to the canonical (Lagrangian and Hamiltonian) formulations of GR, and will use the subject of cosmological perturbations as the principal application. We begin with a study of the Einstein-Hilbert action, and then proceed to consider the more complex issue of the Hamiltonian formulation. The full Hamiltonian of GR will be constructed in the next lecture. The Hamiltonian will then be used to describe the possible initial conditions in GR and the time evolution of spacetimes.

Reading:

- MTW §§21.1–21.3.

II. THE LAGRANGIAN OF GENERAL RELATIVITY

The first step in a canonical treatment of GR is to construct an action that yields Einstein's equations as the Euler-Lagrange equations. In order to do this, we need both an equation for the action and a set of variables that can be varied. In our case, the set of variables will be the fields $g_{\mu\nu}(x^\alpha)$. We may specify either the covariant or contravariant metric; often you will see $g^{\mu\nu}$ written, but of course these contain the same information.

The derivation of the action from a set of equations of motion is very hard, not always possible, and there is no systematic way to do it. We therefore will begin by guessing the action and showing that it gives the right answer. We want our choice to be gauge invariant and "local" in the sense that it will lend itself to being the integral of a Lagrangian; that is, we will guess something like

$$S = \int \mathcal{L}\, d^{(4)}V, \tag{1}$$

where $\mathcal{L}$ is a scalar *Lagrange density* and $d^{(4)}V$ is the element of 4-volume. We thus need both a scalar and the 4-volume element.

A. Setup

The 4-volume element is easiest: we recall that in a locally Minkowskian coordinate system $\{x^{\alpha'}\}_{\alpha'=0}^3$, the volume element is $d^{(4)}V = dx^{0'}\,dx^{1'}\,dx^{2'}\,dx^{3'}$. If there is a positive-determinant (i.e. conserving the handedness of the Jacobian) Jacobian $J^{\alpha'}{}_\beta = \partial x^{\alpha'}/\partial x^\beta$ that transforms this to a general coordinate system $\{x^\beta\}_{\beta=0}^3$, we have

$$d^{(4)}V = dx^{0'}\,dx^{1'}\,dx^{2'}\,dx^{3'} = (\det J^{\alpha'}{}_\beta)\,dx^0\,dx^1\,dx^2\,dx^3 \equiv (\det J^{\alpha'}{}_\beta)\,d^4x. \tag{2}$$

It turns out however that the metric tensor in the general coordinate system is

$$g_{\gamma\delta} = J^{\alpha'}{}_\gamma J^{\beta'}{}_\delta \eta_{\alpha'\beta'}, \tag{3}$$

or in matrix language $\mathbf{g} = \mathbf{J}^{\rm T}\boldsymbol{\eta}\mathbf{J}$. If we define g to be the determinant of the 4×4 matrix $\mathbf{g}$, we then have $g = -(\det\mathbf{J})^2$, so it follows that $\det\mathbf{J} = \sqrt{-g}$. We thus see that the 4-volume element is

$$d^{(4)}V = \sqrt{-g}\,d^4x. \tag{4}$$

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Not so straightforward is guessing the scalar Lagrange density $\mathcal{L}$. In general this has both terms associated with GR and with the matter:

$$\mathcal{L} = \mathcal{L}_{\rm GR} + \mathcal{L}_{\rm matter}, \tag{5}$$

with $\mathcal{L}_{\rm matter} = 0$ for the vacuum. In terms of finding the Euler-Lagrange equations, we can then take the functional derivative

$$\frac{\delta S}{\delta g_{\mu\nu}} = \frac{\delta S_{\rm GR}}{\delta g_{\mu\nu}} + \frac{\delta S_{\rm matter}}{\delta g_{\mu\nu}} = 0. \tag{6}$$

This suggests that we are on the right track: the first term in this equation contains only geometry, the second term contains matter, and they are equal (aside from a sign).

A technical aside: We define $\mathcal{L}$ only for symmetric $g_{\mu\nu}$ since this is the only case that makes sense. Therefore only the symmetric part of $\delta S/\delta g_{\mu\nu}$ is defined, with the antisymmetric part undetermined. We will simply set the antisymmetric part to be equal to zero. This is equivalent to formally defining S for general $g_{\mu\nu}$ in such a way that $S[g_{\mu\nu}] = S[g_{\nu\mu}]$; if we do so then variations of the action with respect to the antisymmetric part of the metric automatically vanish when the derivative is taken at symmetric $g_{\mu\nu}$.

B. The GR terms

Our goal here is to guess the GR piece, which should be a property of only the spacetime. The simplest scalar that we could write down is of course a constant, but this can't be right since a Lagrangian that (in vacuum) depends only on $g_{\mu\nu}$ and not on any derivatives cannot have interesting dynamics. (In fact, if we write down $\mathcal{L}_{\rm GR} = k_2 \neq 0$ then there are no solutions to the Euler-Lagrange equations at all!) We therefore try the next simplest option:

$$\mathcal{L}_{\rm GR} = k_1 R + k_2, \tag{7}$$

where R is the Ricci scalar, and k_1 and k_2 are constants. This will turn out to be the right answer. We see that:

$$\frac{\delta S_{\rm GR}}{\delta g_{\mu\nu}} = k_1 \frac{\delta}{\delta g_{\mu\nu}} \int R\sqrt{-g}\, d^4x + k_2 \frac{\delta}{\delta g_{\mu\nu}} \int \sqrt{-g}\, d^4x. \tag{8}$$

There are two functional derivatives in Eq. (8). The second one is straightforward: since the integral contains no derivatives of $g_{\mu\nu}$, we see that

$$\frac{\delta}{\delta g_{\mu\nu}} \int \sqrt{-g}\, d^4x = \frac{\partial}{\partial g_{\mu\nu}} \sqrt{-g}. \tag{9}$$

We now recall the general rule of derivatives of matrix determinants,

$$\frac{\partial \det \mathbf{A}}{\partial A_{ij}} = (\det \mathbf{A})[\mathbf{A}^{-1}]_{ji}. \tag{10}$$

(Recall the expansion of the determinant across the ith row and Cramer's rule.) Using this, we simplify the partial derivative to

$$\frac{\delta}{\delta g_{\mu\nu}} \int \sqrt{-g}\, d^4x = \frac{\partial}{\partial g_{\mu\nu}} \sqrt{-g} = \frac{-1}{2\sqrt{-g}}(g)\, g^{\nu\mu} = \frac{1}{2} g^{\mu\nu}\sqrt{-g}. \tag{11}$$

This is a tensor multiplied by $\sqrt{-g}$, which is required by general covariance: remember that what Eq. (11) is saying is that the general variation of the integral is

$$\delta \int \sqrt{-g}\, d^4x = \int \left(\frac{1}{2} g^{\mu\nu}\right) \delta g_{\mu\nu} \sqrt{-g}\, d^4x. \tag{12}$$

Clearly the object in parentheses needs to be a tensor.

The determination of the first functional derivative in Eq. (8) by brute force is much harder given the messy dependence of R on $g_{\mu\nu}$. We know however that it will be of the form

$$\frac{\delta}{\delta g_{\mu\nu}} \int R\sqrt{-g}\, d^4x = H^{\mu\nu}\sqrt{-g} \tag{13}$$

for some symmetric tensor $H^{\mu\nu}$. Much effort can be saved by examining the properties of this tensor.

First, we see that R depends on the metric components $g_{\mu\nu}$, their derivatives $g_{\mu\nu,\alpha}$, and their second derivatives $g_{\mu\nu,\alpha\beta}$. Furthermore every term contains exactly 2 derivatives: it may be linear in the second derivative with no dependence on the first derivative, or quadratic in the first derivative with no dependence on the second derivative. It follows that the same is true of $R\sqrt{-g}$. Therefore the functional derivative in Eq. (13) is of the form

$$\frac{\delta}{\delta g_{\mu\nu}}\int \mathcal{U}^{\alpha\beta\gamma\delta\epsilon\zeta}[g_{\mu\nu}]g_{\alpha\beta,\gamma}g_{\delta\epsilon,\zeta}\,d^4x + \frac{\delta}{\delta g_{\mu\nu}}\int \mathcal{V}^{\alpha\beta\gamma\delta}[g_{\mu\nu}]g_{\gamma\delta,\alpha\beta}\,d^4x, \tag{14}$$

where $\mathcal{U}$ and $\mathcal{V}$ are functions. If we integrate by parts on the $\mathcal{V}$ term, then the ∂_β derivative moves onto $\mathcal{V}$ and gives us a term that is quadratic in the first derivatives of $g_{\mu\nu}$ with no second derivatives. Thus, integration by parts allows us to eliminate terms of the $\mathcal{V}$ type and leave us with only terms of the $\mathcal{U}$ type. We may further symmetrize $\mathcal{U}$ under the interchange of $\alpha\beta\gamma \leftrightarrow \delta\epsilon\zeta$ indices. Putting in the coordinate labels and integrating by parts gives

$$\begin{aligned}
\frac{\delta}{\delta g_{\mu\nu}(y^\sigma)}\int R\sqrt{-g}\,d^4x &= \frac{\delta}{\delta g_{\mu\nu}(y^\sigma)}\int \mathcal{U}^{\alpha\beta\gamma\delta\epsilon\zeta}[g_{\mu\nu}(x^\sigma)]g_{\alpha\beta,\gamma}(x^\sigma)g_{\delta\epsilon,\zeta}(x^\sigma)\,d^4x \\
&= \int \frac{\delta\mathcal{U}^{\alpha\beta\gamma\delta\epsilon\zeta}[g_{\mu\nu}(x^\sigma)]}{\delta g_{\mu\nu}(y^\sigma)}g_{\alpha\beta,\gamma}(x^\sigma)g_{\delta\epsilon,\zeta}(x^\sigma)\,d^4x \\
&\quad +2\int \mathcal{U}^{\alpha\beta\gamma\delta\epsilon\zeta}[g_{\mu\nu}(x^\sigma)]\frac{\delta g_{\alpha\beta,\gamma}(x^\sigma)}{\delta g_{\mu\nu}(y^\sigma)}g_{\delta\epsilon,\zeta}(x^\sigma)\,d^4x \\
&= \int \frac{\delta\mathcal{U}^{\alpha\beta\gamma\delta\epsilon\zeta}[g_{\mu\nu}(x^\sigma)]}{\delta g_{\mu\nu}(y^\sigma)}g_{\alpha\beta,\gamma}(x^\sigma)g_{\delta\epsilon,\zeta}(x^\sigma)\,d^4x \\
&\quad -2\int \frac{\partial}{\partial x^\gamma}\left[\mathcal{U}^{\alpha\beta\gamma\delta\epsilon\zeta}[g_{\mu\nu}(x^\sigma)]g_{\delta\epsilon,\zeta}(x^\sigma)\right]\frac{\delta g_{\alpha\beta}(x^\sigma)}{\delta g_{\mu\nu}(y^\sigma)}\,d^4x \\
&= \frac{\partial\mathcal{U}^{\alpha\beta\gamma\delta\epsilon\zeta}}{\partial g_{\mu\nu}}g_{\alpha\beta,\gamma}g_{\delta\epsilon,\zeta}(y^\sigma) - 2\frac{\partial}{\partial x^\gamma}\left[\mathcal{U}^{\mu\nu\gamma\delta\epsilon\zeta}g_{\delta\epsilon,\zeta}(x^\sigma)\right](y^\sigma).
\end{aligned} \tag{15}$$

We thus see that $H^{\mu\nu}$ also depends on the metric components $g_{\mu\nu}$, their derivatives $g_{\mu\nu,\alpha}$, and their second derivatives $g_{\mu\nu,\alpha\beta}$. Furthermore every term contains exactly 2 derivatives: it may be linear in the second derivative with no dependence on the first derivative, or quadratic in the first derivative with no dependence on the second derivative.

This remarkable result heavily restricts the possible forms of $H^{\mu\nu}$. Since we may go to a local Lorentz frame and choose Riemann normal coordinates in which the metric derivatives vanish and the second derivatives of the metric are determined by the Riemann tensor, it follows that $H^{\mu\nu}$ is constructed entirely from the metric and Riemann tensors. Further, the requirement of being at most linear in the second derivative implies that the Riemann tensor occurs at most to linear order: we thus have

$$H^{\mu\nu} = c_1 R^{\mu\nu} + c_2 R g^{\mu\nu}, \tag{16}$$

where c_1, c_2 are constants. (The need for exactly 2 derivatives eliminates a possible $c_3 g^{\mu\nu}$ term.) Obtaining this form is remarkably constraining!

But we can do even better. Note that the action $\int R\sqrt{-g}\,d^4x$ is invariant under gauge transformations that leave the boundary fixed, i.e. if we impose a gauge transformation given by vector field $\boldsymbol{\xi}$, it follows that the action remains invariant under the metric perturbation $\delta g_{\mu\nu} = 2\xi_{(\mu;\nu)}$. Therefore we see that

$$2\int H^{\mu\nu}\xi_{(\mu;\nu)}\sqrt{-g}\,d^4x = 0. \tag{17}$$

We may of course drop the 2. Noting that $H^{\mu\nu}$ is symmetric so we can drop the parentheses, and then performing integration by parts, we see that

$$\int (H^{\mu\nu}\xi_\mu)_{;\nu}\sqrt{-g}\,d^4x - \int H^{\mu\nu}{}_{;\nu}\xi_\mu\sqrt{-g}\,d^4x = 0. \tag{18}$$

The first term here vanishes: it is a surface integral by Stokes's theorem. If you don't like this kind of argument, recall that for any vector v^ν we have

$$v^\nu{}_{;\nu}\sqrt{-g} = v^\nu{}_{,\nu}\sqrt{-g} + \Gamma^\nu_{\nu\beta}v^\beta\sqrt{-g} = v^\nu{}_{,\nu}\sqrt{-g} + (\partial_\beta \ln\sqrt{-g})v^\beta\sqrt{-g} = (v^\nu\sqrt{-g})_{,\nu}, \tag{19}$$

so one can directly see this is a total derivative. Therefore since this is true for general $\boldsymbol{\xi}$ we must have $H^{\mu\nu}{}_{;\nu} = 0$. This only works for general metrics if $c_2 = -\frac{1}{2}c_1$ (recall the Bianchi identity, which tells us that $G^{\mu\nu}{}_{;\nu} = 0$ but that this does not hold for other linear combinations of $R^{\mu\nu} + bRg^{\mu\nu}$). Thus from Eq. (16) we see that

$$H^{\mu\nu} = c_1 G^{\mu\nu}. \tag{20}$$

The appearance of $G^{\mu\nu}$ is promising if we are going to eventually try to recover Einstein's equations. But note the reason why it appeared: it was a consequence of gauge invariance! This contrasts with the approach from 1st term where it was required by local conservation of energy-momentum $T^{\mu\nu}{}_{\nu} = 0$. The two are of course closely related, since the local conservation of energy-momentum is the "conservation law" associated with the symmetry of gauge transformations.

To complete our job, we must find the numerical value of c_1. This can be done with a specific example (**any** nontrivial example will do). Consider the 4-dimensional manifold $S^2 \times T^2$, where the sphere has radius 1 and the torus is compactified with length ℓ in one direction and length 1 in the other. That is, we take the range $0 \le t < 1$, $0 \le z < 1$, and use a standard Minkowskian-signature metric:

$$ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2 + \ell^2\, dz^2 - dt^2. \tag{21}$$

Then the Riemann tensor has $R_{\theta\phi\theta\phi} = 1$ and the other components (not trivially related to $R_{\theta\phi\theta\phi}$ via index permutations) vanish. We have $R = 2$ and total volume $4\pi\ell$, and so

$$\int R\sqrt{-g}\, d^4x = (2)(4\pi\ell) = 8\pi\ell. \tag{22}$$

Thus $\delta\int R\sqrt{-g}\, d^4x = 8\pi\,\delta\ell$. This must be compared with, from Eq. (13),

$$8\pi\,\delta\ell = \int H^{\mu\nu}\delta g_{\mu\nu}\sqrt{-g}\, d^4x = c_1\int G^{\mu\nu}\delta g_{\mu\nu}\sqrt{-g}\, d^4x. \tag{23}$$

Since $\delta g_{zz} = \delta(\ell^2) = 2\ell\,\delta\ell$ and the other components of $\delta g_{\mu\nu}$ vanish, it follows that:

$$8\pi\,\delta\ell = 2c_1\ell\,\delta\ell\int G^{zz}\sqrt{-g}\, d^4x. \tag{24}$$

Now inspection shows that $R^{zz} = 0$ (no Riemann tensor component containing z can be nonvanishing since no metric components containing z are nonconstant and no components depend on z), so we have

$$G^{zz} = -\frac{1}{2}Rg^{zz} = -\frac{1}{2}(2)(\ell^{-2}) = -\ell^{-2}. \tag{25}$$

Multiplying by the volume then gives

$$8\pi\,\delta\ell = 2c_1\ell\,\delta\ell\,(-L^{-2})(4\pi\ell) = -8\pi c_1\,\delta\ell \tag{26}$$

and hence $c_1 = -1$. Putting this all together gives $H^{\mu\nu} = -G^{\mu\nu}$ and hence

$$\frac{\delta S_{\rm GR}}{\delta g_{\mu\nu}} = \left(-k_1 G^{\mu\nu} + \frac{1}{2}k_2 g^{\mu\nu}\right)\sqrt{-g}. \tag{27}$$

The vacuum Einstein equation $G^{\mu\nu} = 0$ is recovered if $k_2 = 0$; so you can probably guess that k_2 is going to be related to the cosmological constant. We will return to this issue after we determine the value of k_1.

III. THE MATTER TERMS

In order to complete our investigation of the Lagrangian formulation of GR, we must turn next to the matter terms: what is $\delta S_{\rm matter}/\delta g_{\mu\nu}$? Clearly we want it to be somehow related to the stress-energy tensor, but we need to find (i) the correct coefficient and (ii) show that this identification is correct. To do both of these, we first consider a particular case – that of a swarm of particles – and show that its stress-energy tensor agrees with $\delta S_{\rm matter}/\delta g_{\mu\nu}$ times $-\frac{1}{2}$. We then show that $\delta S_{\rm matter}/\delta g_{\mu\nu}$ also agrees with the canonical definition of energy.

A. The swarm of particles

Consider a system with a suite of particles $\{A\}$ each of mass μ_A following some set of trajectories $x^\mu(\sigma|A)$ where A is a particle index and σ is a coordinate on the world line. The action for such particles is

$$S = -\sum_A \mu_A \int d\tau = -\sum_A \mu_A \int \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma}}\, d\sigma. \tag{28}$$

The variation of S is

$$\begin{aligned}\delta S &= -\sum_A \mu_A \int \frac{-1}{2\sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma}}}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma}\delta g_{\mu\nu}\, d\sigma \\ &= \frac{1}{2}\sum_A \mu_A \int \frac{1}{d\tau/d\sigma}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\left(\frac{d\tau}{d\sigma}\right)^2 \delta g_{\mu\nu}\, d\sigma \\ &= \frac{1}{2}\sum_A \mu_A \int u^\mu u^\nu \delta g_{\mu\nu}\, d\tau. \end{aligned} \tag{29}$$

Therefore the functional derivative is

$$\frac{\delta S_{\rm matter}}{\delta g_{\mu\nu}(y^\alpha)} = \frac{1}{2}\sum_A \mu_A \int u^\mu u^\nu \delta^{(4)}[y^\alpha - x^\alpha(\tau|A)]\, d\tau. \tag{30}$$

The right-hand side is $\frac{1}{2}$ of the stress-energy tensor at y^α, aside from a missing factor of $\sqrt{-g}$ since the coordinate delta function appears here but the physical volume delta function $(-g)^{-1/2}\delta^{(4)}[y^\alpha - x^\alpha(\tau|A)]$ appears in $T^{\mu\nu}$. We thus learn that

$$\frac{\delta S_{\rm matter}}{\delta g_{\mu\nu}(y^\alpha)} = \frac{1}{2}T^{\mu\nu}(y^\alpha)\sqrt{-g}. \tag{31}$$

This is often taken as the relativistic definition of the stress-energy tensor.

You might object that I didn't prove Eq. (31) for general matter fields. This is a good objection. All I have done is prove a particular case (and you should be able to show easily that the gauge invariance of the matter action implies that the relativistic stress-energy is divergence-free, ${T^{\mu\nu}}_{;\nu} = 0$). A more fundamental objection is that we never defined e.g. the "energy density" in this class, so it is hard to prove that the energy density equals something. I won't completely solve this problem now, **except** to prove that the relativistic stress-energy tensor agrees with the canonical notion of energy density you learned in undergraduate physics.

B. Canonical definition of energy

We now investigate the canonical definition of energy of some system of particles, fields, etc. in an enclosed box of 3-volume V in Minkowski space and in the rest frame of the box. If you don't like closed boxes, then you may think of this exercise with periodic boundary conditions in the spatial dimensions so as to achieve some total 3-volume V. We will work in the usual Minkowski coordinates. Our job is to prove that the integral

$$\mathcal{E}(t) \equiv 2\int \frac{\delta S_{\rm matter}}{\delta g_{00}}\, d^3x \tag{32}$$

obtained as the integral of the relativistic energy density is in fact the usual notion of "energy" from classical mechanics. Since this integral is on a 3-surface of constant t, we see that the **computation** of $\mathcal{E}$ depends on t, although the conservation of relativistic stress-energy shows that it is actually constant.

To do this, let's imagine that the system has some set of N degrees of freedom $\{q_A\}_{A=1}^N$ and a Lagrangian $L(q_A, \dot q_A)$. If the system contains electromagnetic fields etc. then maybe $N = \infty$ but we are physicists and that doesn't bother us. We can then see that the action, measured from time t_0 to time t_2, is

$$S_{\rm matter} = \int_{t_0}^{t_2} L(q_A, \dot q_A)\, dt. \tag{33}$$

Now let's imagine that between some time t_1 and $t_1 + \Delta t$, we change the time-time part of the metric tensor so that instead of $g_{00} = -1$ we have $g_{00} = -1 + \epsilon$ (at all spatial coordinates), with $|\epsilon| \ll 1$ and Δt small compared to the shortest dynamical timescale of the system. It follows that the change of the action is

$$\delta S = \epsilon\,\Delta t \int \frac{\delta S_{\rm matter}}{\delta g_{00}}\, d^3x \bigg|_{t_1} = \frac{1}{2}\epsilon\,\Delta t\,\mathcal{E}(t_1). \tag{34}$$

We may alternatively compute the action by noting that between t_1 and $t_1 + \Delta t$, the time coordinate is stretched by a factor of $1 - \frac{1}{2}\epsilon$, so that during this interval the proper time experienced by the system is only $(1 - \frac{1}{2}\epsilon)\Delta t$. Moreover, the time derivatives $\dot{q}_A$ are sped up by a factor of $1 + \frac{1}{2}\epsilon$. Then the matter action changes by

$$\begin{aligned}
\delta S_{\rm matter} &= \int_{t_1}^{t_1+\Delta t} L\left(q_A, \left(1 + \frac{1}{2}\epsilon\right)\dot{q}_A\right)\,\left(1 - \frac{1}{2}\epsilon\right) dt - \int_{t_1}^{t_1+\Delta t} L(q_A, \dot{q}_A)\, dt \\
&= \Delta t\,\left(\frac{1}{2}\epsilon \frac{\partial L}{\partial \dot{q}_A}\dot{q}_A - \frac{1}{2}\epsilon L\right)\bigg|_{t_1} \\
&= \frac{1}{2}\epsilon\,\Delta t\,\left(\frac{\partial L}{\partial \dot{q}_A}\dot{q}_A - L\right)\bigg|_{t_1}.
\end{aligned} \tag{35}$$

Comparison to Eq. (34) shows that

$$\mathcal{E}(t_1) = \left(\frac{\partial L}{\partial \dot{q}_A}\dot{q}_A - L\right)\bigg|_{t_1}. \tag{36}$$

Thus $\mathcal{E}$ is the Hamiltonian, i.e. the usual definition of the energy.

A good exercise (slightly harder) is to do the same proof for the total momentum rather than the energy.

IV. THE FULL ACTION

We are now ready to write down the field equation for GR: it is $\delta S/\delta g_{\mu\nu} = 0$ where S is the total action (GR + matter). This gives us

$$-k_1 G^{\mu\nu} + \frac{1}{2}k_2 g^{\mu\nu} + \frac{1}{2}T^{\mu\nu} = 0, \tag{37}$$

or – solving for $8\pi T^{\mu\nu}$ – we find

$$16\pi k_1 G^{\mu\nu} - 8\pi k_2 g^{\mu\nu} = 8\pi T^{\mu\nu}. \tag{38}$$

We thus make the identification that $k_1 = 1/(16\pi)$. Actually it is $1/(16\pi G)$: the choice of k_1 is equivalent to determining the coupling constant of gravity. As $k_1 \rightarrow \infty$ (i.e. it takes infinite energy to bend spacetime), gravity is turned off ($G \rightarrow 0$). This is exactly analogous to the problem in quantum field theory of setting the kinetic term of a particle to infinity: it effectively becomes noninteracting if you switch to the canonical normalization of the kinetic term. In exactly the same sense, $\int R\sqrt{-g}\, d^4x$ is the "kinetic energy" term for gravity.

We further see that $8\pi k_2 = \Lambda$: this term in the action is the cosmological constant. We thus have the GR part of the action:

$$S_{\rm GR} = \int \left(\frac{R}{16\pi} + \frac{\Lambda}{8\pi}\right) \sqrt{-g}\, d^4x. \tag{39}$$

The first term here is the original *Einstein-Hilbert action*. The cosmological constant appears here as an additional term in the action associated with not the curvature but the volume of spacetime: it is for this reason that such a term is expected to arise from zero-point fluctuations.