Lecture XXXII: Thermal history of the Universe – I. Radiation

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(Dated: April 20, 2012)

I. OVERVIEW

Our last problem in homogeneous cosmology is to consider the history of the matter content of the universe. This is really a cosmology rather than "pure GR" subject but it is important to the developments that we will consider next.

Our considerations will begin with the radiation-dominated epoch, the earliest stages at which presently understood physics allows us to say much of anything. We then consider the history of the radiation; we will examine the baryons next.

Reading:

- MTW Ch. 28.

II. THE RADIATION-DOMINATED EPOCH

A. The equation of state of perfectly thermalized plasma

We begin our considerations with the primordial plasma. Here we consider a universe in pure thermal equilibrium at temperature T, and with negligible chemical potentials (i.e. we ignore the matter-antimatter asymmetry, which is good for approximating the expansion history as long as radiation dominates). In such a case, we can add up the total energy density of the universe: it is (integrating over phase space)

$$\rho = \sum_i \frac{g_i}{2\pi^2} \int_0^\infty p^2 \sqrt{p^2+m_i^2}\, f_i(p)\, dp = \sum_i \frac{g_i}{2\pi^2} \int_0^\infty \frac{p^2\sqrt{p^2+m_i^2}\, dp}{e^{\sqrt{p^2+m_i^2}/T} \mp 1}, \tag{1}$$

where the sum is over the possible types of elementary particles. Here the upper sign is used for bosonic degrees of freedom and the lower sign for fermions, and g_i is the number of polarization or spin states (e.g. 2 for the photon). In the intermediate expression, we write $f_i(p)$ for the phase space density of particles at 3-momentum p of type i.

As long as all reactions are fast and the universe expands adiabatically, we may consider the entropy density of the universe to decline as $s \propto a^{-3}$. The entropy density can be obtained from the first law of thermodynamics, $T\,dS = dQ - p\,dV$. If we imagine heating up a vacuum up from absolute zero at constant unit volume, this becomes $T\,ds = d\rho$ or

$$s = \int_0 \frac{d\rho}{T} = \int_0^T \frac{1}{T}\frac{d\rho}{dT}\, dT. \tag{2}$$

Therefore, we may treat the composition of the universe by finding the function $\rho(T)$, then finding the function $s(T)$, and then finding the scale factor corresponding to that temperature via

$$a(T) \propto [s(T)]^{-1/3}. \tag{3}$$

The normalization will have to come from some argument that gives us the present-day temperature of the radiation. Of course we know that – we observe the CMB at 2.725 K – but there is an additional subtle correction (associated with neutrino reactions not being "fast" enough to justify the assumption of adiabaticity) that we will have to come back and fix later.

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The integral for the density, Eq. (1), is in general quite messy to compute, but for the most part two simple limiting cases will suffice: if a particle species has low mass ($m_i \ll T$), then we may write

$$\rho_i = \frac{g_i}{2\pi^2}\int_0^\infty \frac{p^3\,dp}{e^{p/T}\mp 1} = \frac{g_i T^4}{2\pi^2}\int_0^\infty \frac{x^3\,dx}{e^x \mp 1} \tag{4}$$

where $x = p/T$. The integral evaluates to $\pi^4/15$ for bosons ($-$ sign) and to $7\pi^4/120$ for fermions ($+$ sign). It is thus common to write

$$\rho_i = \frac{\pi^2 g_{i\star} T^4}{30}, \tag{5}$$

where $g_{i\star} = g_i$ for a low-mass boson and $g_{i\star} = \frac{7}{8}g_i$ for a low-mass fermion. In the opposite case, $m_i \gg T$, the density is exponentially suppressed by $e^{-m_i/T}$ and we may neglect that particle. The transition in which the integral in Eq. (1) is half of the zero-mass value is typically at $T \sim m_i/3$. We will therefore consider a particle species to be active if $m_i < 3T$. We then write

$$\rho = g_\star \frac{\pi^2 T^4}{30}, \tag{6}$$

where $g_\star$ is the sum of the $g_{i\star}$ for each active species i.

As long as we are far from a transition (the mass of an important species), the entropy density from Eq. (2) is then

$$s = \int_0^T \frac{1}{T}\frac{d}{dT}\left(g_\star \frac{\pi^2 T^4}{30}\right) dT \approx g_\star \int_0^T \frac{1}{T}\frac{d}{dT}\frac{\pi^2 T^4}{30}\,dT = \frac{2\pi^2 g_\star T^3}{15}. \tag{7}$$

Thus we see that as long as $g_\star$ remains constant, $s \propto a^{-3}$ implies that the temperature drops as $T \propto a^{-1}$. When the temperature drops to a few times the mass of a particle, that particle is no longer present in thermal equilibrium (again assuming zero chemical potential – the small asymmetry of particles over antiparticles will become important later) and is said to *annihilate*. The annihilation dumps energy into the other species, such that aT increases and forms a new constant value: in fact it is the combination $as^{1/3} = g_\star^{1/3}aT$ that is constant, and $g_\star$ drops as the universe cools and fewer particles contribute.

B. Expansion history

The Friedmann equation then specifies the expansion history of the universe (assumed spatially flat – at early times this is a good approximation since the radiation density $\propto a^{-4}$ vastly exceeds K/a^2 if a is small enough). Then we find

$$H^2 = \frac{8}{3}\pi\rho = \frac{4\pi^3 g_\star T^4}{45}. \tag{8}$$

Since in radiation domination at constant $g_\star$ we have $H \propto T^2 \propto a^{-2}$, it follows that $a \propto t^{1/2}$ and $H = 1/(2t)$, so

$$t = \frac{1}{2H} = \frac{3\sqrt{5}}{4\pi^{3/2} g_\star^{1/2} T^2} = 1\, g_\star^{-1/2}\left(\frac{1.56\,\mathrm{MeV}}{T}\right)^2\ \mathrm{s}. \tag{9}$$

Thus higher temperatures were reached a shorter time after the Big Bang.

C. The particle zoo

We now go to an epoch when the temperature was a few MeV, of order a tenth of a second after the Big Bang. We will then proceed to examine $g_\star$ at higher and higher temperatures with more particles.

At a few MeV, we have the following particles active:

- *Photons* with 2 polarizations: $g_\star(\gamma) = 2$.
- *Neutrinos* are fermions (factor of $\frac{7}{8}$) and have 3 flavors ($e\mu\tau$) and have antiparticles (ν or $\bar\nu$). Thus $g_\star(\nu) = \frac{7}{8}\times 3\times 2 = 5.25$.

- *Electrons* are fermions, have 2 spin states and can be either e^- or e^+. Thus $g_\star(e) = \frac{7}{8} \times 2 \times 2 = 3.5$.

Therefore at this time the total $g_\star = 10.75$.

As we go to earlier times, additional particles become active.

- At $T \sim 35$ MeV, we add the *muons* – just like the electrons: $g_\star(\mu) = 3.5$. This brings the total to $g_\star = 14.25$.
- At $T \sim 45$ MeV, we add the *pions* – these are spin 0 and have 3 isospin states (π^-, π^0, and π^+), so $g_\star(\pi) = 3$. This brings the total to $g_\star = 17.25$.
- At $T \sim 150$ MeV, the *QCD phase transition* takes place. At temperatures above this, quarks and gluons are free to roam the universe, whereas at lower temperatures the condensate of quark-antiquark pairs in the vacuum makes the vacuum exclude QCD-electric fields and hence binds the quarks and gluons into QCD-neutral objects. Above the QCD phase transition, the pion degrees of freedom disappear (-3). However, many more degrees of freedom come into play: there are 3 flavors of quark (*uds* for *up*, *down*, and *strange*), each a fermion with 3 color states, 2 spin states, and an antiparticle, so we have $g_\star(uds) = \frac{7}{8} \times 3 \times 3 \times 2 \times 2 = 31.5$. We also have 8 gluon color states, each with 2 polarizations, so $g_\star(g) = 16$. The additoin of these brings us to a total $g_\star = 61.75$.
- At $T \sim 500$ MeV, we add the *charm* quark and the *tau* lepton, which contribute 10.5 and 3.5 respectively for a total $g_\star = 75.75$.
- At $T \sim 1.5$ GeV, we add the *bottom* quark for $g_\star(b) = 10.5$, raising us to $g_\star = 86.25$.
- At $T \sim 40$ GeV, we add the heavy electroweak bosons (W$^\pm$ and Z), each with 3 polarization states for a contribution of 9; and the *top* quark for a contribution of 10.5, raising us to $g_\star = 105.75$.

Well, that's all the **known** particles. If you count the Higgs as something we have really good reasons to expect, then the total of all known particles at temperatures of ≥ 40 GeV might rise by 1 to $g_\star = 106.75$.

Are there yet more particles at higher energies? I won't go through all the possibilities here, except for a few comments:

- The *electroweak phase transition* is expected at temperatures of order 100 GeV. Nobody really knows how it works (the LHC will provide some pointers here), but at least in the Standard Model it does not change the number of degrees of freedom. Instead at higher temperature the condensate of Higgs field vanishes, so the W and Z bosons become massless and lose 3 degrees of freedom – but then there are 4 Higgs particles, so $g_\star$ is still 106.75.
- In the Minimal Supersymmetric Standard Model (i.e. imposing supersymmetry and adding only the new particles we need) you would get to $g_\star = 228.75$.

We have used our understanding of physics to work back to only 10^{-10} seconds after the Big Bang. Maybe when you teach GR we can push back further? In the meantime, however, we will push the opposite direction: we will study what happens to the universe at temperatures below an MeV, and in particular determine the distribution of relics (radiation and light elements) produced by a radiation-dominated FRW fireball.

III. THE DECOUPLING OF NEUTRINOS

As the universe cools and becomes less dense, the various particles interact less frequently and at some point they are no longer in equilibrium. The first species of particle to go out of equilibrium is the neutrino, at a temperature of $T \sim 1.7$ MeV. We are interested in the consequences of this for the the expansion history and a vs. T curve of the universe.

The epoch of *neutrino decoupling* occurs when the rate of neutrino interactions Γ_ν becomes comparable to the Hubble rate H. The neutrino interaction rate is dominated by weak interactions between the neutrinos and the electrons/positrons. We thus have

$$\Gamma_\nu = n_e \sigma_{\nu e}, \tag{10}$$

since the neutrinos travel at the speed of light. (We use $\sigma_{\nu e}$ to denote the weak cross section.) We have a number density of electrons given by the blackbody formula, $n_e = 5 \times 10^{31} T_{\rm MeV}^3\,{\rm cm}^{-3}$, where $T_{\rm MeV}$ is the temperature in MeV. The interaction cross section for the weak force increases with energy as $\sigma_{\nu e} \sim 10^{-44} E_{\rm MeV}^2\,{\rm cm}^2$, where $E_{\rm MeV}$ is the neutrino energy in MeV (typically $E \sim 3T$). We thus have

$$\Gamma_\nu \sim 4.5 \times 10^{-12} T_{\rm MeV}^5\,{\rm cm}^{-1} \sim 0.14 T_{\rm MeV}^5\,{\rm s}^{-1}. \tag{11}$$

Using $g_\star = 10.75$ and Eq. (9), we find that

$$\frac{\Gamma_\nu}{H} \sim 2\Gamma_\nu t = 2\left[0.14\times 10^{-2} T_{\rm MeV}^5 {\rm s}^{-1}\right]\left[\frac{1}{\sqrt{10.75}}\left(\frac{1.56\,{\rm MeV}}{T}\right)^2 1\,{\rm s}\right] = 0.2 T_{\rm MeV}^3. \tag{12}$$

This implies that $\Gamma_\nu \sim H$ when $T_{\rm MeV} = 0.2^{-1/3}$ or $T = 1.7\,{\rm MeV}$. Of course, the transition is not actually instantaneous, but this provides a rough guide to when the universe became transparent to neutrinos.

When the neutrinos decouple, we have $g_\star = 10.75$ since at this epoch, the electrons are close to fully relativistic ($T \gg m_e/3$). Thereafter, the various particles continue to cool as $T \propto 1/a$. Therefore, they remain at the same temperature even if the interactions are turned off. This situation continues until the electrons annihilate at $T \sim m_e/3$. The neutrinos remain relativistic and non-interacting, so they continue to cool as $T_\nu \propto 1/a$.

The electron-positron-photon plasma behaves differently: at e^+e^- annihilation, its $g_\star$ drops from 5.5 to 2, i.e. by a factor of $\frac{4}{11}$. Since this annihilation is nearly adiabatic owing to the rapid electromagnetic interactions, we have $T \propto g_\star^{-1/3} a^{-1}$. Therefore, after e^+e^- annihilation, we again have $T \propto 1/a$, but the coefficient is increased by a factor of $(\frac{4}{11})^{-1/3}$. The photon temperature after e^+e^- annihilation thus satisifes

$$T_\gamma = \left(\frac{11}{4}\right)^{1/3} T_\nu, \tag{13}$$

so $T_\nu = 0.71 T_\gamma$. Thus the temperature of the cosmic neutrino background is $1.95a^{-1}\,{\rm K}$.

The energy density of the universe at $T \ll m_e$ is predominately in photons and neutrinos:

$$\rho = 2\frac{\pi^2 T_\gamma^4}{30} + \frac{21}{4}\frac{\pi^2 T_\nu^4}{30} = \left[2 + \frac{21}{4}\left(\frac{4}{11}\right)^{4/3}\right]\frac{\pi^2 T_\gamma^4}{30} = 3.36\frac{\pi^2 T_\gamma^4}{30}. \tag{14}$$

It is therefore conventional to define the "effective number of degrees of freedom" $g_{\star,{\rm eff}} = 3.36$ so that the usual relation between the energy density and the "temperature" (by which we ordinarily mean T_γ) is satisfied.

We now have all the machinery in place to determine the complete relation of T_γ vs. a from $T \sim 100$ GeV to the present day. What is left is to consider the history of the baryonic matter.