Lecture XXXI: The classical tests and density parameters

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I. OVERVIEW

In this section, we will consider the evolution of the matter content of the Universe, and consider the classical tests of cosmology. That is, we will examine the methods of establishing cosmic distances and times, and their dependence on the cosmological parameters.

We won't cover the issue of cosmological perturbations and the tests based on them (e.g. CMB anisotropies, large scale structure) yet.

Reading:

- MTW Ch. 28.

II. LIGHT RAYS AND CONFORMAL TIME

We return to the FRW metric:

$$ds^2 = -dt^2 + a^2(t)[d\chi^2 + D^2(\chi)\,(d\theta^2 + \sin^2\theta\,d\phi^2)], \tag{1}$$

where the distance function is

$$D(\chi) = \begin{cases} \chi & K = 0 \\ K^{-1/2}\sin(K^{1/2}\chi) & K > 0 \\ (-K)^{-1/2}\sinh[(-K)^{1/2}\chi] & K < 0 \end{cases}. \tag{2}$$

In order to determine the trajectories of light rays (a common problem, both observationally and in terms of understanding causal structure), we will introduce the *conformal time* η

$$\eta = \int \frac{dt}{a}, \tag{3}$$

so that

$$ds^2 = a^2(\eta)[-d\eta^2 + d\chi^2 + D^2(\chi)\,(d\theta^2 + \sin^2\theta\,d\phi^2)]. \tag{4}$$

Like the cosmic time t, the conformal time η has no independently defined zeropoint: we choose the integration constant in Eq. (3) based on convenience.

The propagation of light rays in the metric of Eq. (4) is easy to understand. We recall that if λ is the affine parameter for a photon,

$$\frac{dp_\alpha}{d\lambda} = \frac{d(g_{\alpha\beta}p^\beta)}{d\lambda} = g_{\alpha\beta}\frac{dp^\beta}{d\lambda} + g_{\alpha\beta,\gamma}\frac{dx^\gamma}{d\lambda}p^\beta = -g_{\alpha\beta}\Gamma^\beta_{\gamma\delta}p^\gamma p^\delta + g_{\alpha\beta,\gamma}p^\gamma p^\beta = \frac{1}{2}g_{\gamma\delta,\alpha}p^\gamma p^\delta, \tag{5}$$

where in the last equality we expanded the Christoffel symbol. Now in the case of Eq. (4), we have the special circumstance that

$$\frac{\partial g_{\mu\nu}}{\partial\eta} = 2\frac{d\ln a}{d\eta}g_{\mu\nu}. \tag{6}$$

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It follows that for any null curve,

$$\frac{\partial g_{\mu\nu}}{\partial \eta} p^\mu p^\nu = 0, \tag{7}$$

and so Eq. (5) then informs us that $dp_\eta/d\lambda = 0$. Thus we conclude that in FRW cosmology, p_η is conserved. Equivalently, if we go back to the $t\chi\theta\phi$ coordinates, we have

$$p_t = \frac{p_\eta}{a} \propto \frac{1}{a}. \tag{8}$$

Since the energy of a photon measured by a comoving observer is $-p_t$, we conclude that as the Universe expands, the energy of a photon measured in what happens to be the local comoving frame decreases as $E \propto 1/a$. This phenomenon is known as the *cosmological redshift*.

In cosmology the redshift is one of the easiest quantities to measure, since a spectral line from a distant galaxy has a known rest-frame wavelength and the present-day wavelength can be measured. If the distant galaxy and us, the observer, are in a comoving frame, the wavelengths of emission and observation are related by

$$\frac{\lambda_{\rm obs}}{\lambda_{\rm em}} = \frac{a(t_{\rm obs})}{a(t_{\rm em})}. \tag{9}$$

[Here λ denotes wavelength, **not** affine parameter – sorry!] Astronomers conventionally define instead the *redshift*

$$z \equiv \frac{\lambda_{\rm obs}}{\lambda_{\rm em}} - 1 = \frac{a(t_{\rm obs})}{a(t_{\rm em})} - 1. \tag{10}$$

Usually we will also define $a = 1$ at the present day. (The alternative convention if the Universe is open or closed is to set the spatial curvature $K = \pm 1$ and thus define a to be the radius of curvature of the Universe. This definition, common in many early references, is very rare today since all the formulae are discontinuous at zero spatial curvature.) Then it follows that

$$a(t_{\rm em}) = \frac{1}{1+z}. \tag{11}$$

Commonly we will use the redshift of the object and the scale factor $a(t_{\rm em})$ at which radiation reaching us was emitted interchangeably. This is true as long as the emitter and the observer are both comoving. Any deviation from this is called a *peculiar velocity* (precise definition: the 3-velocity of an object relative to a comoving observer). Typical peculiar velocities of galaxies are of order $10^{-3}c$. The peculiar velocity of the Solar System is 370 km/s (measured accurately via the CMB dipole, a purely special relativistic effect). The peculiar velocities of other galaxies are not directly measured, but when they are at cosmological distances ($z \gg 10^{-2}$) they can usually be neglected. (The exception is in the statistics of galaxy clustering, where we try to measure the distance between two far-away objects that are near each other in 3D space.)

III. DISTANCE MEASURES IN COSMOLOGY

An observational cosmologist is immediately faced with an issue – when we see an object, how do we determine its coordinates and the key properties such as its age? Clearly if we place ourselves at the (spatial) origin, then the angular coordinates θ and ϕ are easy to find (we only see light rays with zero angular momentum and hence θ, ϕ are constant). Then a light ray traveling toward us, in order to be null, has

$$\frac{d\chi}{d\eta} = -1. \tag{12}$$

It follows that the redshift is related to the coordinates η and χ via the relation:

$$\chi = \eta_0 - \eta, \qquad z = \frac{1}{a(\eta)} - 1. \tag{13}$$

Here the $_0$ subscript indicates the present day. The coordinate χ is called the *radial comoving distance* to a galaxy: it is the distance that could be measured if, hypothetically, a sequence of rulers at rest relative to comoving observers were laid down at the present day (i.e. on the surface Σ_{t_0}) and stretched end-to-end from us to that galaxy. Obviously, we can't actually measure this.

It is of somewhat greater interest to construct distances that can be measured. For each standard method in astronomy of measuring distance, one can construct a cosmological analogue. We consider the two most interesting cases: standard rulers and standard candles.

A. Standard rulers

A *standard ruler* is an object of known physical size Δs. If it subtends an angle $\Delta\theta$, then we say that its *angular diameter distance* D_A is

$$D_A = \frac{\Delta s}{\Delta\theta}, \tag{14}$$

as this is the distance one infers from its apparent size and simple trigonometry. From the aforementioned metric, we find that

$$D_A = aD(\chi) = \frac{D(\chi)}{1+z}. \tag{15}$$

The challenge is to find an object of known size. Unfortunately, the obvious choices don't work: galaxies can have a range of sizes and they evolve and merge as a function of time, and stars are essentially pointlike at cosmological distances.

A variant on the standard ruler is to take an object whose comoving size ΔX is fixed – i.e. so that its size today is ΔX but where it grows with the Universe in proportion to $\propto a(t)$. Then we define the *comoving angular diameter distance* D_{AC}:

$$D_{AC} = \frac{\Delta X}{\Delta\theta} = D(\chi). \tag{16}$$

This would apply to e.g. features in the galaxy distribution (the main option, which we will discuss later, is the baryon-acoustic oscillation), which thus far have had greater success observationally than the conventional standard ruler idea.

B. Standard candles

A *standard candle* is an object of known luminosity L (units: erg/s). If the observed flux is F (units: erg/cm^2/s), then we say that the *luminosity distance* D_L is

$$D_L = \sqrt{\frac{L}{4\pi F}} \tag{17}$$

(from $L = 4\pi F D_L^2$).

To compute the luminosity distance, imagine a blackbody radiation source of some radius R and temperature T. It emits a luminosity in its rest frame of

$$L = 4\pi R^2 \sigma T^4. \tag{18}$$

When it is observed by us, the solid angle occupied is determined by the angular diameter distance,

$$\Omega = \pi \left(\frac{R}{D_A}\right)^2. \tag{19}$$

However the radiation that we see from the object is no longer a blackbody at temperature T. It has expanded with the Universe and is now a blackbody at temperature $T/(1+z)$. [The argument is the same as for the CMB: photons have their wavelengths stretched by a factor of $1+z$ and are diluted by a factor of $(1+z)^3$, so they retain a blackbody spectrum.] The flux is now

$$F = \frac{1}{\pi}\Omega\sigma T^4 = \frac{R^2\sigma[T/(1+z)]^4}{D_A^2}, \tag{20}$$

where $\frac{1}{\pi}\sigma T^4$ is the flux density of a blackbody seen at normal incidence (units: erg/cm^2/s/sr). We then equate this with

$$F = \frac{L}{4\pi D_L^2} = \frac{R^2\sigma T^4}{D_L^2} \tag{21}$$

to see that

$$D_L = (1+z)^2 D_A = (1+z) D(\chi). \tag{22}$$

The luminosity distance-redshift relation $D_L(z)$ thus contains the same information as the angular or comoving angular diameter distance relations $D_{AC}(z)$ and $D_A(z)$, just with different factors of $1+z$.

The most successful standard candle thus far has been the Type Ia supernovae – this was one of the key pieces of evidence for $\Lambda > 0$.

IV. COSMOLOGICAL PARAMETERS

It's possible to write the present state of the Universe directly in terms of the density of each component, as well as the spatial curvature and expansion rate. In practice, cosmologists usually use a slightly different parameterization.

A. Hubble constant

The first of the usual cosmological parameters is the *Hubble constant* H_0: the present-day value of H. From this one can define the *critical density* $\rho_{\rm cr}$ via

$$\rho_{\rm cr} = \frac{3H_0^2}{8\pi}, \tag{23}$$

which is the density of matter (summed over all forms) that would be required to have a spatially flat cosmology. If $\rho < \rho_{\rm cr}$ then the Universe must be open, and if $\rho > \rho_{\rm cr}$ then it must be closed. The numerical value, in conventional units, is

$$\rho_{\rm cr} = 1.88 \times 10^{-29} h^2 \,{\rm g/cm}^3, \tag{24}$$

where $h = (H_0/100\ {\rm km/s/Mpc})$ is a commonly used shorthand for the Hubble constant. (According to fits to CMB anisotropies from WMAP: $h = 0.710 \pm 0.025$.) Clearly the critical density is very low by everyday standards – but it turns out that the mean density of normal matter is only a few percent of critical!

B. Density parameters

Now the Universe contains many types of matter; for type X, we define the *density parameter*

$$\Omega_X = \frac{\rho_X}{\rho_{\rm cr}}. \tag{25}$$

There may be many contributions:

- *Baryonic matter* (which to an astronomer includes electrons even though they are not really baryons in the particle physics sense) has density parameter $\Omega_{\rm b}$.
- *Dark matter* (whatever it is) has density parameter $\Omega_{\rm dm}$. (Note: sometimes you will see $\Omega_{\rm c}$, where the "c" refers to the dark matter being cold, i.e. with negligible velocity dispersion.) Nonrelativistic matter in total has density parameter $\Omega_{\rm m} = \Omega_{\rm b} + \Omega_{\rm dm}$.
- *Radiation* has density parameter $\Omega_{\rm r}$. This includes both photons (the CMB) and – in cases where mass can be neglected – neutrinos, i.e. $\Omega_{\rm r} = \Omega_\gamma + \Omega_\nu$.
- The cosmological constant has density parameter $\Omega_\Lambda = \Lambda/(8\pi\rho_{\rm cr}) = \Lambda/(3H_0^2)$. [It is conventional for this purpose to put Λ on the right hand side of Einstein's equations.]

[To be exact we should distinguish e.g. the density parameter $\Omega_{\rm m0}$ today from that $\Omega_{\rm m}(t)$ at some other time, since it is not constant. In practice, we write $\Omega_{\rm m}$ to mean the value today unless otherwise specified.]

The total density parameter of everything in the Universe gets the label Ω and determines the geometry. Specifically, the first Friedmann equation evaluated today says

$$\frac{8}{3}\pi\rho_0 = H_0^2 + K, \tag{26}$$

or setting $\rho_0 = \Omega\rho_c$:

$$K = H_0^2(\Omega - 1). \tag{27}$$

It is common convention to define yet another parameter

$$\Omega_K \equiv -\frac{K}{H_0^2}, \tag{28}$$

which is the curvature measured in units of the inverse-squared Hubble length. It is a measure of how much spatial curvature affects the easily accessible regions of the cosmos (out to $z \sim$ a few). Note that sadly it has the **opposite** sign to K: for a closed model $\Omega_K < 0$ and for an open model $\Omega_K > 0$. The advantage of this definition is that one then has all the Ωs add to 1:

$$\Omega_m + \Omega_r + \Omega_\Lambda + \Omega_{\text{anything else out there}} + \Omega_K = 1. \tag{29}$$

In cosmology, one usually quotes the value of Ω_K rather than K. It should be remembered that unlike the other Ωs, Ω_K does not correspond to the actual density of anything – it is most definitely part of the curvature side of Einstein's equation, not the matter. However, in terms of implementation in the Friedmann equation it is convenient, as we see next.

C. The expansion history and distances

We are now ready to write the Friedmann equations in the form most familiar to cosmologists. Let us take the first Friedmann equation,

$$\frac{8}{3}\pi\rho = H^2 + \frac{K}{a^2}. \tag{30}$$

Writing $\rho = \rho_m + \rho_r + \rho_\Lambda$, and recalling the scalings with redshift (i.e. as a^{-3}, a^{-4}, and a^0), we find

$$\frac{8}{3}\pi\rho_{cr}(\Omega_m a^{-3} + \Omega_r a^{-4} + \Omega_\Lambda) = H^2 - \frac{\Omega_K H_0^2}{a^2}. \tag{31}$$

Substituting for ρ_{cr} and solving for H gives

$$H(a) = H_0\sqrt{\mathcal{E}(a)}, \tag{32}$$

where the *energy function* is

$$\mathcal{E}(a) = \Omega_m a^{-3} + \Omega_r a^{-4} + \Omega_\Lambda + \Omega_K a^{-2}. \tag{33}$$

Thus in the expansion history, the curvature acts just like a new type of matter with $w = -\frac{1}{3}$ (**but** the spatial geometry makes it distinguishable).

The time and distance measures can then be obtained by integration. We note that

$$\frac{dt}{da} = \frac{1}{aH} = \frac{1}{H_0 a\sqrt{\mathcal{E}(a)}}, \tag{34}$$

so

$$t = \int \frac{da}{H_0 a\sqrt{\mathcal{E}(a)}}. \tag{35}$$

The conformal time has an additional factor of $1/a$: thus

$$\eta = \int \frac{da}{H_0 a^2\sqrt{\mathcal{E}(a)}}, \tag{36}$$

and the radial comoving distance to an object seen at scale factor a is

$$\chi = -\int_1^a \frac{da}{H_0 a^2\sqrt{\mathcal{E}(a)}}. \tag{37}$$

V. DISTANCES IN THE FRW UNIVERSE – EXAMPLES

Having defined the various distances, our next objective is to understand how they relate to the density and curvature of the Universe. This subject underlies much of modern observational cosmology.

A. Einstein-de Sitter model

The simplest model of the cosmos we consider has only matter and is flat: this is the *Einstein-de Sitter model*, $\Omega_{\rm m} = 1$ and $\Omega_{\rm K} = \Omega_\Lambda = 0$. This model has energy function

$$\mathcal{E}(a) = a^{-3}. \tag{38}$$

We then see that the age of the Universe is

$$t = \frac{1}{H_0}\int a^{1/2}\,da = \frac{2a^{3/2}}{3H_0}, \tag{39}$$

so the present-day age ($a = 1$) is $t_0 = 2/(3H_0)$.

The conformal time is

$$\eta = \frac{1}{H_0}\int a^{-1/2}\,da = \frac{2a^{1/2}}{H_0}. \tag{40}$$

Of key interest to us is that in the Einstein-de Sitter universe, the conformal time has a lower limit – at the Big Bang $\eta = 0$ – but no upper limit: the universe lives forever and $\eta \to \infty$. The consequence is that any two comoving observers are in causal contact with each other if they wait long enough.

The distance measures can now be constructed: we have

$$\chi = \eta(a = 1) - \eta(a) = \frac{2}{H_0}(1 - a^{1/2}) = \frac{2}{H_0}\left(1 - \frac{1}{\sqrt{1+z}}\right). \tag{41}$$

The comoving angular diameter distance D_{AC} is the same as χ since the model is spatially flat. The angular diameter and luminosity distances are different by factors of $1 + z$:

$$D_A = \frac{2}{H_0(1+z)}\left(1 - \frac{1}{\sqrt{1+z}}\right) \quad \text{and} \quad D_L = \frac{2}{H_0}(1+z)\left(1 - \frac{1}{\sqrt{1+z}}\right). \tag{42}$$

Here D_L is a monotonically increasing function of z, but a peculiar property of cosmology (including not just the Einstein-de Sitter model but the ΛCDM model as well!) is that D_A reaches a maximum at $z = 1.25$. Beyond some redshift, more distant objects appear **closer**, and as $z \to \infty$ we have $D_A \to 0$. Thus the spots in the CMB, which have physical sizes of order 10^5 light-years, appear to have angular sizes of tens of arcminutes, even though they are only a few times larger (in physical scale) than galaxies (which are arcsecond-size at $z \sim 1$).

However, the Einstein-de Sitter model does **not** describe our Universe: it cannot reproduce the supernova luminosity distance results or galaxy clustering measurements. We thus turn our attention to more complicated models: the open model (which provides a reasonable description of local galaxy clustering data but has serious problems with CMB anisotropies and also with the supernovae), and then the ΛCDM model (which is consistent with all of the data on the global structure of our Universe).

B. Open and closed models with $\Lambda = 0$

Next we consider the open universe model. This has only matter and no cosmological constant: $\Omega_{\rm m} < 1$ and $\Lambda = 0$, but is curved, $\Omega_{\rm K} = 1 - \Omega_{\rm m}$. The same formulae describe the closed model with $\Omega_{\rm m} > 1$ via analytic continuation. Both of these are now of historical interest.

In the open model, we have

$$\mathcal{E}(a) = a^{-2}[\Omega_{\rm m}(a^{-1} - 1) + 1]. \tag{43}$$

The cosmic age is

$$t = \frac{1}{H_0} \int \frac{da}{\sqrt{\Omega_{\rm m}(a^{-1} - 1) + 1}}. \tag{44}$$

There is nothing especially remarkable about this except for its limits: $t \approx a^{3/2}$ at small a and $t \approx a$ at large a (open case). Thus this cosmological model expands forever at a rate da/dt that approaches a constant.

The closed case is a bit different: there we find that $\mathcal{E}(a) = 0$ at some maximum value of a,

$$a_{\rm max} = \frac{1}{1 - \Omega_{\rm m}^{-1}}. \tag{45}$$

The Friedmann equation then tells us that $H = 0$ when a reaches $a_{\rm max}$. Study of the integrand in Eq. (44) shows that this is reached in a finite time, since the integrand has only a square-root divergence in the denominator, i.e. as we approach $a_{\rm max}$ we have $dt/da \propto (a_{\rm max} - a)^{-1/2}$. Thereafter, the closed universe turns around and recollapses, in a time-reverse of its expansion. The universe ends in a *Big Crunch* in a finite amount of time given by

$$\begin{aligned} t_{\rm Bang\to Crunch} &= 2H_0^{-1} \int_0^{a_{\rm max}} \frac{da}{\sqrt{\Omega_{\rm m}(a^{-1} - 1) + 1}} \\ &= 2H_0^{-1} a_{\rm max} \int_0^1 \frac{dx}{\sqrt{\Omega_{\rm m}(a_{\rm max}^{-1} x^{-1} - 1) + 1}} \\ &= 2H_0^{-1} a_{\rm max} \int_0^1 \frac{dx}{\sqrt{\Omega_{\rm m}[(1 - \Omega_{\rm m}^{-1}) x^{-1} - 1] + 1}} \\ &= 2H_0^{-1} a_{\rm max} \int_0^1 \frac{dx}{\sqrt{(\Omega_{\rm m} - 1)(x^{-1} - 1)}} \\ &= \frac{2\Omega_{\rm m}}{H_0(\Omega_{\rm m} - 1)^{3/2}} \int_0^1 \frac{dx}{\sqrt{x^{-1} - 1}} \\ &= \frac{\pi\Omega_{\rm m}}{H_0(\Omega_{\rm m} - 1)^{3/2}}. \end{aligned} \tag{46}$$

where we defined $a = a_{\rm max} x$, and the last integral is solved by the substitution $x = \cos^2\alpha$.

Of greater interest both theoretically and observationally is the conformal time, which has an additional factor of a^{-1} relative to Eq. (44):

$$\eta = \frac{1}{H_0} \int \frac{da}{a\sqrt{\Omega_{\rm m}(a^{-1} - 1) + 1}}. \tag{47}$$

This is most easily evaluated by switching the integration variable to z:

$$\eta = -\frac{1}{H_0} \int \frac{dz}{(1+z)\sqrt{\Omega_{\rm m} z + 1}}. \tag{48}$$

If we define the new variable $y = 1 + \Omega_{\rm m} z$, then

$$\eta = -\frac{1}{H_0} \int \frac{dy}{(\Omega_{\rm m} - 1 + y)\sqrt{y}}. \tag{49}$$

Finally, if we define $y = (\Omega_{\rm m} - 1)\beta^2$, then we find

$$\eta = -\frac{2}{H_0}(\Omega_{\rm m} - 1)^{1/2} \int \frac{d\beta}{1 + \beta^2} = -\frac{2}{H_0}(\Omega_{\rm m} - 1)^{1/2} \arctan\beta = -2K^{-1/2} \arctan\beta = -2R \arctan\beta, \tag{50}$$

where R is the radius of curvature of the universe. Now in the case of a closed universe, we find that y runs from ∞ at the Big Bang to 0 at turnaround ($a = a_{\rm max}$), so correspondingly β goes from ∞ to 0. It follows that the conformal time changes by $\eta_{\rm turnaround} - \eta_{\rm Bang} = \pi R$. Equivalently, in the lifetime of the universe from Bang to Crunch, the conformal time increases by $2\pi R$. We thus conclude that **a closed matter-dominated universe lives exactly**

long enough for a light ray to go all the way around once between the Big Bang and the Big Crunch. Even if we lived in such a universe, we couldn't see ourselves.

Quite the opposite is true for the open model: here y runs from ∞ at the Big Bang to $1-\Omega_\mathrm{m}$ as $a\to\infty$. In this case, we set $y=(1-\Omega_\mathrm{m})\gamma^2$ and find in place of Eq. (50):

$$\eta = -\frac{2}{H_0}(1-\Omega_\mathrm{m})^{1/2}\int\frac{d\gamma}{\gamma^2-1} = -\frac{2}{H_0}(1-\Omega_\mathrm{m})^{1/2}\,\mathrm{arccoth}\,\gamma = -2(-K)^{-1/2}\,\mathrm{arccoth}\,\gamma. \tag{51}$$

Since arccoth $\infty = 0$ and arccoth $1=\infty$, it follows that the open model has infinite conformal age in the future. Thus – just like for the Einstein-de Sitter case – any two observers are in causal contact if they wait long enough.

The distance measures can be computed in either of these cases (and the formulae are related by analytic continuation). From Eq. (50), the radial comoving distance in a closed model is

$$\begin{aligned}
\chi &= \eta(z=0)-\eta(z)\\
&= \eta(\beta=\sqrt{1/(\Omega_\mathrm{m}-1)})-\eta(\beta=\sqrt{(1+\Omega_\mathrm{m}z)/(\Omega_\mathrm{m}-1)})\\
&= -2K^{-1/2}\left[\arctan\sqrt{\frac{1}{\Omega_\mathrm{m}-1}}-\arctan\sqrt{\frac{1+\Omega_\mathrm{m}z}{\Omega_\mathrm{m}-1}}\right]\\
&= 2K^{-1/2}\left[\arctan\sqrt{\frac{1+\Omega_\mathrm{m}z}{\Omega_\mathrm{m}-1}}-\arctan\sqrt{\frac{1}{\Omega_\mathrm{m}-1}}\right].
\end{aligned} \tag{52}$$

This looks messy but recall that what we need is $\sin(K^{1/2}\chi)$. We thus use

$$\sin\vartheta = \frac{2\tan(\vartheta/2)}{1+\tan^2(\vartheta/2)}, \tag{53}$$

and then recall that if $\vartheta/2=\arctan A-\arctan B$ we may use the tangent difference formula:

$$\tan\frac{\vartheta}{2}=\frac{A-B}{1+AB}, \tag{54}$$

so that

$$\sin\vartheta = \frac{2(A-B)/(1+AB)}{1+(A-B)^2/(1+AB)^2} = \frac{2(A-B)(1+AB)}{(1+AB)^2+(A-B)^2} = \frac{2(A-B)(1+AB)}{(1+A^2)(1+B^2)}. \tag{55}$$

Applied in the case of $A=\sqrt{(1+\Omega_\mathrm{m}z)/(\Omega_\mathrm{m}-1)}$ and $B=\sqrt{1/(\Omega_\mathrm{m}-1)}$ gives

$$\begin{aligned}
\sin\vartheta &= \frac{2(\sqrt{1+\Omega_\mathrm{m}z}-1)[1+\sqrt{1+\Omega_\mathrm{m}z}/(\Omega_\mathrm{m}-1)]/\sqrt{\Omega_\mathrm{m}-1}}{[1+(1+\Omega_\mathrm{m}z)/(\Omega_\mathrm{m}-1)][1+1/(\Omega_\mathrm{m}-1)]}\\
&= (\Omega_\mathrm{m}-1)^{1/2}\frac{2(\sqrt{1+\Omega_\mathrm{m}z}-1)(\Omega_\mathrm{m}+\sqrt{1+\Omega_\mathrm{m}z}-1)}{\Omega_\mathrm{m}^2(1+z)}.
\end{aligned} \tag{56}$$

Now recalling that $D_L=(1+z)K^{-1/2}\sin(K^{1/2}\chi)$, we conclude that

$$D_L = 2H_0^{-1}\frac{(\sqrt{1+\Omega_\mathrm{m}z}-1)(\Omega_\mathrm{m}+\sqrt{1+\Omega_\mathrm{m}z}-1)}{\Omega_\mathrm{m}^2}. \tag{57}$$

This equation is valid for either open or closed models.

A few aspects of Eq. (57) are of direct observational interest. One is the behavior at low z: if we Tayor expand,

$$D_L = H_0^{-1}\left[z+\left(\frac{1}{2}-\frac{1}{4}\Omega_\mathrm{m}\right)z^2+...\right]. \tag{58}$$

Thus the luminosity distance is linearly related to redshift for nearby objects, but when $z\sim\mathcal{O}(1)$ the next-order terms are important. The luminosity distance increases faster in open universes than in Einstein-de Sitter, making $D_L(z)$ one of the classical methods for measuring Ω_m. This was in fact the principal goal of the supernova efforts that discovered Λ, except that they found that the luminosity distance increased even faster than Eq. (58) for $\Lambda=0$.

The behavior at high z is different: Eq. (57) gives

$$D_L\to\frac{2z}{\Omega_\mathrm{m}H_0}\quad\text{or}\quad D_{AC}\to\frac{2}{\Omega_\mathrm{m}H_0}\quad(z\gg 1). \tag{59}$$

C. ΛCDM model

The next case that we will consider is the ΛCDM model (where CDM stands for *cold dark matter*, which is the second greatest component of the cosmic pie). This model, unlike the others we have encountered, is in agreement with the data on the global structure of the Universe.

In this model, the universe is spatially flat ($K = 0$) with $\Omega_\mathrm{m} + \Omega_\Lambda = 1$. The energy function is

$$\mathcal{E}(a) = \Omega_\mathrm{m} a^{-3} + 1 - \Omega_\mathrm{m}, \tag{60}$$

and the age of the universe is

$$t = \frac{1}{H_0} \int \frac{da}{a\sqrt{\Omega_\mathrm{m}(a^{-3} - 1) + 1}}. \tag{61}$$

The limiting cases are $t \propto a^{3/2}$ at small a (i.e. during the matter-dominated phase) and $t \propto H_0^{-1}(1 - \Omega_\mathrm{m})^{-1/2} \ln a$ at large a (i.e. in the Λ-dominated phase). Thus we see that in the far future, the universe expands exponentially (a is an exponential function of t).

We may also find the radial distance to an object via

$$\chi = -\frac{1}{H_0} \int_1^a \frac{da}{a^2 \sqrt{\mathcal{E}(a)}} = \frac{1}{H_0} \int_0^z \frac{dz}{\sqrt{\Omega_\mathrm{m}(1+z)^3 + 1 - \Omega_\mathrm{m}}}. \tag{62}$$

The Taylor expansion of the argument of the square root is $1 + 3\Omega_\mathrm{m} z + ..$, so we find

$$\chi = \frac{1}{H_0} \int_0^z \left(1 - \frac{3}{2}\Omega_\mathrm{m} z + ...\right) dz = \frac{1}{H_0} \left[z - \frac{3}{4}\Omega_\mathrm{m} z^2 + ...\right], \tag{63}$$

and so the luminosity distance $(1 + z)\chi$ (recall: this is a spatially flat model) is

$$D_L = \frac{1}{H_0} \left[z + \left(1 - \frac{3}{4}\Omega_\mathrm{m}\right) z^2 + ...\right]. \tag{64}$$

In Eq. (58) for the open model, we were limited by the condition of positive matter density to have a quadratic coefficient (of z^2) no larger than $\frac{1}{2}$. However, the Λ model can accommodate a coefficient as large as 1. Thus, the evidence from supernovae for a strong increase of the luminosity distance with redshift rules out the open $\Lambda = 0$ model but is consistent with Λ. Other combinations, such as models with both Λ and spatial curvature, are allowed but the magnitude of Ω_K is limited by other data (notably the CMB anisotropies).

The behavior of the conformal time is more fundamental: we have

$$\eta = \frac{1}{H_0} \int \frac{da}{a^2\sqrt{\Omega_\mathrm{m} a^{-3} + 1 - \Omega_\mathrm{m}}}. \tag{65}$$

If we define $a_\star = [\Omega_\mathrm{m}/(1 - \Omega_\mathrm{m})]^{1/3}$, then defining $a = a_\star/u$ we may write this as

$$\eta = -\frac{1}{H_0 a_\star \sqrt{1 - \Omega_\mathrm{m}}} \int \frac{du}{\sqrt{u^3 + 1}}. \tag{66}$$

Of particular interest is that η has a finite range: the change in conformal time from the Big Bang to the indefinite future is

$$\begin{aligned} \Delta\eta &= -\frac{1}{H_0 a_\star \sqrt{1 - \Omega_\mathrm{m}}} \int_\infty^0 \frac{du}{\sqrt{u^3 + 1}} \\ &= \frac{1}{H_0 \Omega_\mathrm{m}^{1/3} (1 - \Omega_\mathrm{m})^{1/6}} \int_0^\infty \frac{du}{\sqrt{u^3 + 1}} \\ &= \frac{2.804}{H_0 \Omega_\mathrm{m}^{1/3} (1 - \Omega_\mathrm{m})^{1/6}}. \end{aligned} \tag{67}$$

The fact that the integral converges at the lower limit is nothing new: it merely says that we can at present see only galaxies out to some maximum (comoving) distance. But the convergence at the upper limit says the universe has a finite conformal lifetime. That is, while the physical lifetime of the universe is infinite – a comoving observer will see the universe go on an infinite amount of proper time – there are objects farther away from us than $\Delta\eta$ that we can never see and they can never see us. Moreover, even the galaxies we can see now at high redshift are at distance χ exceeding the remaining conformal lifetime of the universe $\eta(\infty)-\eta_0$. We can never send any signals to these galaxies: if you shoot your laser pointer at them, the photons travel only a finite comoving distance between now and $t=\infty$, and rather than reaching those galaxies the photons are destined to become stuck in the exponentially expanding and diluting intergalactic medium.

In the Λ-dominated universe, then, the future is that any systems not gravitationally bound to us by perturbations (i.e. more than ~ 10 Mpc away) will become permanently causally disconnected from us. There is a last signal that we can send to such systems, and there is a last signal that they can send to us (it is very much like the event horizon of a black hole, except that this time it is symmetrical between the two observers). The effect of Λ on observables thus far may be subtle, but – if Λ is really the correct explanation for cosmic acceleration – it condemns us to the loneliest of fates!