Lecture XXX: Dynamics of cosmic expansion

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I. OVERVIEW

In this lecture, we will consider the dynamics of the homogeneous, isotropic (FRW) universe.
Reading:

- MTW Ch. 27.

II. THE FRIEDMANN EQUATIONS

In order to write the Einstein equations for an FRW universe, we first need the Ricci tensor. A comoving observer with 4-velocity $\boldsymbol{u}$ and spatial basis $\boldsymbol{e}_{\hat{i}}$ ($\hat{i} = 1, 2, 3$; we will take $\boldsymbol{e}_{\hat{1}} = a^{-1}\boldsymbol{e}_\chi$, $\boldsymbol{e}_{\hat{2}} = a^{-1}\chi^{-1}\boldsymbol{e}_\theta$, and $\boldsymbol{e}_{\hat{3}} = a^{-1}\chi^{-1}\csc\theta\,\boldsymbol{e}_\phi$) can break the Ricci tensor into 10 components:

- 1 time-time component $R_{\mu\nu}u^\mu u^\nu$;
- 3 time-space components, $R_{\mu\nu}u^\mu(e_{\hat{i}})^\nu$; and
- 6 space-space components, $R_{\mu\nu}(e_{\hat{i}})^\mu(e_{\hat{j}})^\nu$.

The time-time component is a 3-scalar (invariant under rotations of the observer's 3 spatial basis vectors), R_{tt}. The time-space components form a vector and by isotropy must be zero. The space-space components form a symmetric tensor and are restricted by isotropy to have the form

$$R_{\mu\nu}(e_{\hat{i}})^\mu(e_{\hat{j}})^\nu = L\delta_{\hat{i}\hat{j}}. \tag{1}$$

Thus there are two nontrivial components here: R_{tt} and $L = a^{-2}R_{\chi\chi}$.

The computation of the Ricci tensor is straightforward and leads to the result:

$$\begin{aligned} R_{tt} &= -3\frac{\ddot{a}}{a} \quad \text{and} \\ L = \frac{R_{\chi\chi}}{a^2} &= \frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{K}{a^2}. \end{aligned} \tag{2}$$

In the orthonormal basis $\{\boldsymbol{u} = \boldsymbol{e}_{\hat{0}}, \boldsymbol{e}_{\hat{1}}, \boldsymbol{e}_{\hat{2}}, \boldsymbol{e}_{\hat{3}}\}$, we have

$$R_{\hat{\alpha}\hat{\beta}} = \begin{pmatrix} R_{tt} & 0 & 0 & 0 \\ 0 & L & 0 & 0 \\ 0 & 0 & L & 0 \\ 0 & 0 & 0 & L \end{pmatrix}. \tag{3}$$

Then the Einstein tensor is

$$G_{\hat{\alpha}\hat{\beta}} = \begin{pmatrix} \frac{1}{2}R_{tt} + \frac{3}{2}L & 0 & 0 & 0 \\ 0 & \frac{1}{2}R_{tt} - \frac{1}{2}L & 0 & 0 \\ 0 & 0 & \frac{1}{2}R_{tt} - \frac{1}{2}L & 0 \\ 0 & 0 & 0 & \frac{1}{2}R_{tt} - \frac{1}{2}L \end{pmatrix}. \tag{4}$$

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The Einstein equations then equate this to $8\pi T_{\hat\alpha\hat\beta}$. The diagonal nature of $T_{\hat\alpha\hat\beta}$ forces the stress-energy tensor to be that of a fluid – with, in the rest frame of a comoving observer, zero momentum and an isotropic pressure:

$$\begin{aligned} 8\pi G\rho &= \frac{1}{2}R_{tt} + \frac{3}{2}L \quad \text{and} \\ 8\pi G p &= \frac{1}{2}R_{tt} - \frac{1}{2}L. \end{aligned} \tag{5}$$

Note that this does **not** imply that the universe must consist of an actual fluid (we will find some counterexamples later on). Anything with a stress-energy tensor of this form is acceptable.

Normally we take a linear combination of the basic equations, Eq. (5). The first equation divided by 3 gives

$$\frac{8}{3}\pi G\rho = \left(\frac{\dot a}{a}\right)^2 + \frac{K}{a^2}. \tag{6}$$

If we take $-\frac{1}{6}$ times the ρ equation plus $-\frac{1}{2}$ times the p equation, we get

$$-\frac{4}{3}\pi G(\rho + 3p) = \frac{\ddot a}{a}. \tag{7}$$

These are called the *Friedmann equations*.

III. THE CONTINUITY EQUATION

Continuity provides a strong constraint on the possible behavior of matter in cosmology. We derive the continuity equation in the FRW metric in two ways – using GR machinery, and using conventional thermodynamics. It is not a priori obvious that these must agree, since the definition of "pressure" used in the stress-energy tensor (rate of transport of momentum) is not the same as the definition of "pressure" in thermodynamics (a derivative of energy with respect to volume). In fact in a broad range of circumstances (see the homework) these can be shown from their definitions to be equivalent.

A. Relativistic derivation

We next consider the implications of the continuity equation, $T_\mu{}^\nu{}_{;\nu} = 0$. The stress-energy tensor can be written as

$$T_\mu{}^\nu = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}. \tag{8}$$

Its divergence is

$$0 = T_\mu{}^\nu{}_{;\nu} = T_\mu{}^\nu{}_{,\nu} + \Gamma^\nu_{\nu\alpha} T_\mu{}^\alpha - \Gamma^\alpha_{\mu\nu} T_\alpha{}^\nu. \tag{9}$$

Since $T_\mu{}^\nu{}_{;\nu}$ is a 4-vector, symmetry considerations (isotropy) imply that only the $\mu = t$ component is nontrivial. The Christoffel symbols that we need are

$$\Gamma^t_{tt} = 0 \quad \text{and} \quad \Gamma^i_{tj} = H\delta^i_j. \tag{10}$$

From here we may write Eq. (9) as

$$0 = -\dot\rho - 3H\rho - 3Hp, \tag{11}$$

or

$$\dot\rho = -3H(\rho + p). \tag{12}$$

This is the usual form of the continuity equation.

B. Thermodynamic derivation

The continuity equation can also be derived on thermodynamic grounds. Recall from the first law of thermodynamics that the change in internal energy of a system dU is

$$dU = dQ - p\,dV, \tag{13}$$

where dQ is the heat input and dV is the volume. In our context a parcel of the cosmos has no heat input (where would that come from?) so we have simply $dU = -p\,dV$. Now the energy of this parcel is $U = \rho V$, so

$$-p\,dV = dU = d(\rho V) = \rho\,dV + V\,d\rho. \tag{14}$$

Therefore dividing by V we have

$$-p\,d\ln V = \rho\,d\ln V + d\rho. \tag{15}$$

Solving for $d\rho$ and dividing by the differential of time, we get

$$\dot\rho = -(\rho + p)\frac{d}{dt}\ln V. \tag{16}$$

Since $V \propto a^3$, the last derivative is $3H$, and we recover Eq. (12).

IV. EXAMPLE – CONSTANT PRESSURE-TO-DENSITY RATIO

Often in cosmology we will consider cases where the pressure-to-density ratio is constant; this is often called the *equation of state* w:

$$w = \frac{p}{\rho}. \tag{17}$$

An obvious example is *nonrelativistic matter* (often called "dust" but I will avoid this term as it has another meaning to astronomers!), which has $|p| \ll \rho$ or $|w| \ll 1$. Another is *radiation* (any gas of particles with negligible mass such as photons or, at high enough redshift, neutrinos) with $w = \frac{1}{3}$.

A less obvious example is that the cosmological constant may also be considered as something with constant equation of state $w = -1$. Remember that in the presence of a cosmological constant, Einstein's equation reads

$$G^{\mu\nu} - \Lambda g^{\mu\nu} = 8\pi T^{\mu\nu}. \tag{18}$$

If we move Λ to the right-hand side, we can see that its effect is equivalent to adding something with a stress-energy tensor

$$T^{(\Lambda)\,\mu\nu} = \frac{\Lambda}{8\pi} g^{\mu\nu}, \tag{19}$$

that is with $\rho = \Lambda/(8\pi)$ and $p = -\Lambda/(8\pi)$. The cosmological constant thus acts like a medium with $w = -1$. Whether at a fundamental level theoretical physics should distinguish between a $w = -1$ contribution to the stress-energy tensor (arising from the quantum physics of the vacuum) and a cosmological constant (arising from the gravitational sector) is open to debate – but as far as observables in cosmology are concerned we may treat them as one and the same.

The continuity equation, Eq. (12), then tells us that

$$\dot\rho = -3(1 + w)H\rho, \tag{20}$$

and since $H = d(\ln a)/dt$ we find

$$\rho \propto a^{-3(1+w)}. \tag{21}$$

Thus:

- The density of nonrelativistic matter scales as $\rho \propto a^{-3}$ (as one would expect by diluting particles in accordance with the cosmological volume).

- The density of radiation scales as $\rho \propto a^{-4}$. This is appropriate for redshifting of radiation: if we stretch the radiation by a factor of 2 in every direction then not only does the number density of photons go down by a factor of 8, the energy per photon decreases by a factor of 2, so the overall energy density goes down by a factor of 16.
- The cosmological constant ($w = -1$) has a density $\rho =$ constant, hence its name.

If the Universe is spatially flat ($K = 0$) and is composed of matter with constant w, then its expansion history is quite simple to determine. Using the Friedmann equation, we find

$$\frac{8}{3}\pi G\rho_0 a^{-3(1+w)} = \left(\frac{\dot{a}}{a}\right)^2 = \frac{(da/dt)^2}{a^2}, \tag{22}$$

where ρ_0 is the present-day density. Solving for dt gives

$$\left(\frac{8}{3}\pi G\rho_0\right)^{1/2} dt = a^{3(1+w)/2-1}\, da. \tag{23}$$

We may then integrate to find

$$\left(\frac{8}{3}\pi G\rho_0\right)^{1/2} t = \frac{2}{3(1+w)} a^{3(1+w)/2}, \tag{24}$$

where we set $t = 0$ at the Big Bang ($a = 0$). The expansion history of the Universe is then

$$a = \left[\frac{3(1+w)}{2}\right]^{2/[3(1+w)]} \left(\frac{8}{3}\pi G\rho_0\right)^{1/[3(1+w)]} t^{2/[3(1+w)]}. \tag{25}$$

We thus see the proportionality: $a \propto t^{2/[3(1+w)]}$. If the Universe is matter-dominated, then $w = 0$ and $a \propto t^{2/3}$. If the Universe is radiation-dominated, as it was for the first 10^4 years of its life, then $a \propto t^{1/2}$.

The cosmological constant is an exception to all this: if $w = -1$ then the above integrals are invalid. Instead, the Friedmann equation tells us that $H = \sqrt{\Lambda/3}$, so the solution for the expansion of the Universe is an exponential rather than a power law:

$$a \propto e^{\sqrt{\Lambda/3}\,t}. \tag{26}$$

Note that $w = -1$ does not allow for a Big Bang. Moreover, all times are equivalent in this spacetime (an observer sees the same curvature tensor at any t) so the spacetime is invariant to time translations. In fact it is invariant under boosts as well, as you will prove on the homework. This special spacetime,

$$ds^2 = -dt^2 + e^{2\sqrt{\Lambda/3}\,t}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2], \tag{27}$$

is called *de Sitter spacetime*.

[Note: if $\Lambda < 0$ then there are no spatially flat cosmologies allowed. There are other solutions, including a static spacetime with negative spatial curvature and $\dot{a} = 0$ known as *anti-de Sitter spacetime* (AdS) but they don't describe our universe.]