Lecture XXV: Reissner-Nordstrøm black holes

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I. OVERVIEW

The Schwarzschild black hole is the simplest black hole solution. However, more complicated types are possible. The most astrophysically realistic case is that of a rotating, neutral black hole. (Interstellar space is conducting; a black hole charged to the extent that its electromagnetic field affects its structure should quickly discharge.) But the case of a charged spherical black hole exhibits many of the same properties and is easier to study, so we consider it first.

Reading:

- MTW Ch. 33.
- If you're interested in learning more about the global structure, see Hawking & Ellis, *The Large-Scale Structure of Spacetime*.
- The great details of perturbations of the Reissner-Nordstrøm metric, and what goes wrong at the inner horizon, can be found in Chandrasekhar, *The Mathematical Theory of Black Holes*.

II. THE REISSNER-NORDSTRØM METRIC

The *Reissner-Nordstrøm metric* was introduced on the homework as the solution to Einstein's equations for a spherically symmetric system with a radial electric field and zero 4-current density.

The metric of interest to us is

$$ds^2 = -\left(1 - 2\frac{M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \frac{dr^2}{1 - 2M/r + Q^2/r^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{1}$$

It is spherically symmetric and time-independent, but has $T_{\mu\nu} \neq 0$. If one constructs a local orthonormal basis,

$$\boldsymbol{e}_{\hat t} = \left(1 - 2\frac{M}{r} + \frac{Q^2}{r^2}\right)^{-1/2} \boldsymbol{e}_t, \quad \boldsymbol{e}_{\hat r} = \left(1 - 2\frac{M}{r} + \frac{Q^2}{r^2}\right)^{1/2} \boldsymbol{e}_r, \quad \boldsymbol{e}_{\hat\theta} = \frac{1}{r}\boldsymbol{e}_\theta, \quad \text{and} \quad \boldsymbol{e}_{\hat\phi} = \frac{1}{r\sin\theta}\boldsymbol{e}_\phi, \tag{2}$$

then an observer who is stationary with respect to the coordinate system (4-velocity $\boldsymbol{u} = \boldsymbol{e}_{\hat t}$) sees an outward-pointed electric field with magnitude $E_{\hat r} = Q/r^2$. The electromagnetic field strength is

$$F_{tr} = -\frac{Q}{r^2}, \quad F_{rt} = \frac{Q}{r^2}, \quad \text{other components zero.} \tag{3}$$

[Note that due to an accident in the form of the metric, namely that it is diagonal and $g_{tt} = 1/g_{rr}$, we find that in a local orthonormal frame aligned with the coordinate axes, $F_{\hat t\hat r} = F_{tr}$ – this is not at all true in general!] This is derivable from a 1-form potential, $\boldsymbol{F} = \boldsymbol{dA}$, with potential

$$A_t = -\frac{Q}{r}, \quad \text{other components zero.} \tag{4}$$

The behavior of the Reissner-Nordstrøm solution depends on the relation of the charge Q to the mass M. Defining the function $f(r) = 1 - 2M/r + Q^2/r^2$, we see that:

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- If $|Q| < M$, then $f(r)$ has two zeroes at $r_\pm = M \pm \sqrt{M^2 - Q^2}$. The exterior looks qualitatively similar to the Schwarzschild spacetime. In particular, if we take $Q \to 0$, then the outermost coordinate singularity occurs at $r_+ = 2M$.
- If $|Q| \geq M$, then everything is well-behaved at positive r [$f(r) > 0$]. The central singularity at $r = 0$ (which, like in Schwarzschild, is a physical singularity) is visible to the outside world as a *naked singularity.*

We are interested in two key questions related to the Reissner-Nordstrøm metric:

- For $|Q| < M$, what is the global structure of the solution? Where are the event horizons, and for a collapsing star which regions are physical? What happens to an observer who falls in? [This is analogous to our study of Schwarzschild.]
- Can a black hole be "charged up" to $|Q| \geq M$, e.g. by feeding it many particles with the same-sign charge? By any such process is it possible that we might see a naked singularity?

We will answer the first question. For the second question – it seems likely that the answer is no (at least we will see that simplistic attempts to set up such conditions fail) but as yet the issue of naked singularities remains unresolved.

III. THE GLOBAL STRUCTURE OF REISSNER-NORDSTRØM BLACK HOLES

We begin our study of the global structure of the Reissner-Nordstrøm hole by considering the case of $|Q| < M$. We then follow the analysis of the Schwarzschild case; the only difference is the function $f(r)$, which now has a quadratic term. In particular, we may convert the metric to the form

$$ds^2 = f(r)(-dt^2 + dr_\star^2) + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{5}$$

by defining a radial coordinate

$$r_\star = \int \frac{dr}{f(r)}. \tag{6}$$

This time, we have

$$f(r) = 1 - 2\frac{M}{r} + \frac{Q^2}{r^2} = \left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right), \tag{7}$$

with $0 < r_- < r_+$. The key difference from Schwarzschild will be that $f(r)$ has two zeroes instead of one – inside r_- it goes positive again! The partial fractions separation gives

$$\begin{aligned}
r_\star &= \int \frac{dr}{(1 - r_+/r)(1 - r_-/r)} \\
&= \frac{1}{r_+ - r_-}\int \left(\frac{r^2}{r - r_+} - \frac{r^2}{r - r_-}\right) dr \\
&= \frac{1}{r_+ - r_-}\int \left(r_+ - r_- + \frac{r_+^2}{r - r_+} - \frac{r_-^2}{r - r_-}\right) dr \\
&= r + \frac{r_+^2}{r_+ - r_-}\ln\frac{r - r_+}{2M} - \frac{r_-^2}{r_+ - r_-}\ln\frac{r - r_-}{2M}.
\end{aligned} \tag{8}$$

[Remember that the integration constant is arbitrary.]

Our next step, again, is to define coordinates $\tilde{U} = t - r_\star$ and $\tilde{V} = t + r_\star$, with which

$$ds^2 = -f(r)d\tilde{U}\, d\tilde{V} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{9}$$

An observer is confined to a trajectory that moves to the upper-right in the $(\tilde{U}, \tilde{V})$ plane. The exterior of the hole is $(\tilde{U}, \tilde{V}) \in \mathbb{R}^2$.

Equation (9) has the same pathology as its Schwarzschild analogue, namely that at $r = r_\pm$ the metric tensor components go to zero, i.e. the coordinate system becomes singular such that points that are really neighbors of each other are far-separated in the $(\tilde{U},\tilde{V})$ plane. We may cure this deficiency in the same way, i.e. by defining new functions $\tilde{u}(\tilde{U})$ and $\tilde{v}(\tilde{V})$. We would like these functions to be well-behaved across the horizon, i.e. we want

$$\frac{1}{f(r)}\frac{d\tilde{u}}{d\tilde{U}}\frac{d\tilde{v}}{d\tilde{V}} = \text{finite at } r = r_\pm. \tag{10}$$

That is, we want a simple zero in either $d\tilde{u}/d\tilde{U}$ or $d\tilde{v}/d\tilde{V}$ at $r = r_\pm$ – i.e. when $r_\star$ goes to $\pm\infty$. We thus try the option

$$\tilde{u} = -e^{-\tilde{U}/(4K)}, \quad \tilde{v} = e^{\tilde{V}/(4K)}, \tag{11}$$

where we take

$$K = \frac{r_+^2}{2(r_+ - r_-)} \tag{12}$$

so that the behavior at $r \approx r_+$ will be well-behaved. [Note that for a Schwarzschild hole ($Q = 0$) we have $K = M$, and for $Q \neq 0$ we have $K > M$.] We also see that

$$\frac{r_-^2}{r_+ - r_-} = \frac{r_+^2}{r_+ - r_-} - (r_+ + r_-) = 2K - 2M. \tag{13}$$

Then we have

$$ds^2 = 16K^2\frac{f(r)}{\tilde{u}\tilde{v}}d\tilde{u}\,d\tilde{v} + r^2(d\theta^2 + \sin^2\theta\,d\phi^2). \tag{14}$$

The exterior of the hole ($r > r_+$) is at $\tilde{u} < 0$, $\tilde{v} > 0$, just as for Schwarzschild. This equation implicitly contains r. This is the solution of

$$e^{r_\star/(2K)} = -\tilde{u}\tilde{v}. \tag{15}$$

So now $\tilde{u} = 0$ or $\tilde{v} = 0$ corresponds to $r = r_+$. Exponentiating Eq. (8), we find that

$$e^{r_\star/(2K)} = e^{r/(2K)}\frac{r - r_+}{2M}\left(\frac{r - r_-}{2M}\right)^{-K/M+1}. \tag{16}$$

Using that $f(r) = (r - r_+)(r - r_-)/r^2$, we thus see that the metric reduces to

$$ds^2 = -\frac{16K^2(2M)^{2-K/M}}{4r^2}(r - r_-)^{K/M}e^{-r/(2K)}\,d\tilde{u}\,d\tilde{v} + r^2(d\theta^2 + \sin^2\theta\,d\phi^2). \tag{17}$$

The original time coordinate is now

$$t = 2K\ln\frac{-\tilde{v}}{\tilde{u}}. \tag{18}$$

With the form Eq. (17), we have performed a similar feat to that that we accomplished for Schwarzschild – the exterior of the hole is now the upper-left quadrant of the plane, and the future horizon is at $\tilde{u} = 0$, $\tilde{v} > 0$. Moreover the spacetime diagram is now symmetric under 180° rotation, $(\tilde{u},\tilde{v}) \to (-\tilde{u},-\tilde{v})$. But unlike Schwarzschild, there is no singularity in the upper-right quadrant. The system remains perfectly well-behaved for any $(\tilde{u},\tilde{v}) \in \mathbb{R}^2$. This time, as $\tilde{u}\tilde{v} \to +\infty$, we have $e^{r_\star/(2K)} \to -\infty$ and hence $r \to r_-$. What happens at the surface $r = r_-$? Another transformation is needed to discern this!

Fortunately, we already have such a transformation: we let $\tilde{u} = \tan^j\xi$, $\tilde{v} = \tan^j\eta$ for some positive j (to be determined). The plane $(\tilde{u},\tilde{v}) \in \mathbb{R}^2$ maps to the square $(\xi,\eta) \in (-\frac{\pi}{2},\frac{\pi}{2})^2$, and the $r_\star$ coordinate now satisfies

$$e^{r_\star/(2K)} = -(\tan\xi\tan\eta)^j. \tag{19}$$

The metric is now

$$ds^2 = -\frac{16K^2(2M)^{2-K/M}}{4r^2}(r - r_-)^{K/M}e^{-r/(2K)}j^2\tan^{j-1}\xi\tan^{j-1}\eta\sec^2\xi\sec^2\eta\,d\xi\,d\eta + r^2(d\theta^2 + \sin^2\theta\,d\phi^2). \tag{20}$$

As e.g. $\eta \to \pi/2$, we find that $e^{r_\star/(2K)} \propto -(\pi/2-\eta)^j$ and so $(r-r_-)^{-K/M+1} \propto (\pi/2-\eta)^j$. The coefficient of $d\xi\, d\eta$ in the line element then scales as

$$g_{\xi\eta} \propto (r-r_-)^{K/M} \tan^{j-1}\eta \sec^2\eta \propto \left(\frac{\pi}{2}-\eta\right)^{[j/(-K/M+1)]K/M+j-1+2}. \tag{21}$$

Thus if

$$j = \frac{K}{M} - 1 = \frac{2M^2 - Q^2 + 2M\sqrt{M^2-Q^2}}{2M\sqrt{M^2-Q^2}} - 1 = \frac{2M^2-Q^2}{2M\sqrt{M^2-Q^2}}, \tag{22}$$

which is $j > 0$ for $0 < |Q| < M$, then Eq. (20) is well-behaved at $r = r_-$. The $r = r_-$ surface is called an *inner event horizon*, since again it separates observers who cannot send signals back into the domain $(0, \frac{\pi}{2})^2$.

The behavior of the new coordinate system when either ξ or η exceeds $\pi/2$ is most easily seen by noting that

$$\tan\xi \tan\eta = [-e^{r_\star/(2K)}]^{1/j} = e^{r/(2jK)} \left(\frac{r_+ - r}{2M}\right)^{1/j} \left(\frac{2M}{r-r_-}\right). \tag{23}$$

So once ξ or η moves to $> \pi/2$ this is negative, implying $r < r_-$. Note however that $r = 0$ – the true singularity – occurs at a finite negative value of $\tan\xi\tan\eta$. Note that the singularity slopes to the upper-right – it is timelike, not spacelike as in the case of Schwarzschild. This means an observer could in principle see it (except see below).

The continuation of this process leads to Fig. 34.4 of MTW. A particle crossing the inner horizon at $\eta = \pi/2$ may reach the singularity, or choose instead to continue on to the upper-right to $\xi = \pi/2$, at which point $\tan\xi = \infty$ and $r = r_-$. So the analytic continuation of Reissner-Nordstrøm keeps repeating itself – if ξ and η are increased by π, then the same original situation occurs.

Which portions of the spacetime are physical? Clearly if the black hole formed from a collapsing charged star, only the "outside" part of it. But this allows an observer to reach the inner event horizon, $r = r_-$. In the vacuum solution, just like the outer horizon $r = r_+$, nothing happens there. But there is a difference – an observer reaching $r = r_-$ ($\eta = \pi/2$) can see the entire history of the outside Universe, including events that occurred in the infinite future as seen by someone outside. Thus, all objects in the outside Universe are seen with infinite blue-shift! In any realistic situation then, this leads to infinite stress-energy tensor from perturbations to the exact Reissner-Nordstrøm solution. Thus it is believed that as far as the validity of GR is concerned, an infalling observer reaches their end at $r = r_-$.

IV. CHARGING A BLACK HOLE

We now come to the problem of charging a Reissner-Nordstrøm black hole – in particular, starting with a Schwarzschild hole, can we inject particles of the same charge to charge it to $Q = M$? To do so we have to consider the orbits of charged particles. Without loss of generality we will consider a positively charged black hole.

A. Orbits of small charged particles

We consider a test particle of mass μ and charge q, and suppose it orbits within the equatorial plane (spherical symmetry allows us to do this). For any Killing field $\boldsymbol{\xi}$, recall that the quantity $\boldsymbol{\xi} \cdot (\mu \boldsymbol{u} + q\boldsymbol{A})$ is conserved. The timelike Killing field $\xi^\alpha = (1,0,0,0)$ leads to the conservation of energy

$$\mathcal{E} = -\mu u_t - eA_t = -\mu u_t + \frac{qQ}{r}, \tag{24}$$

and the azimuthal Killing field leads to conservation of angular momentum,

$$\mathcal{L} = \mu u_\phi. \tag{25}$$

B. Allowed regions

Outside the black hole, at $r > r_+$, the conditions $g^{\alpha\beta}u_\alpha u_\beta = -1$ and that $(u_r)^2 \ge 0$ leads to an allowed region:

$$f(r)(u_r)^2 = -1 + \frac{1}{f(r)}(u_t)^2 - r^{-2}u_\phi^2 \ge 0, \tag{26}$$

or

$$-1+\frac{(\mathcal{E}-qQ/r)^2}{\mu^2(1-2M/r+Q^2/r^2)}-\frac{\mathcal{L}^2}{\mu^2r^2}\geq 0. \tag{27}$$

We may isolate $\mathcal{E}$:

$$\left(\frac{\mathcal{E}}{\mu}-\frac{qQ}{\mu r}\right)^2\geq\left(1-2\frac{M}{r}+\frac{Q^2}{r^2}\right)\left(1+\frac{\mathcal{L}^2}{\mu^2r^2}\right). \tag{28}$$

There are thus two allowed regions for the energy – one continuously connected to ∞ (corresponding to particles moving forward in time), and one continuously connected to $-\infty$ (corresponding to particles moving backward in time, which we do not consider). Taking the physical case, we have

$$\frac{\mathcal{E}}{\mu}\geq\frac{qQ}{\mu r}+\sqrt{\left(1-2\frac{M}{r}+\frac{Q^2}{r^2}\right)\left(1+\frac{\mathcal{L}^2}{\mu^2r^2}\right)}. \tag{29}$$

In the large-r (Newtonian) limit, taking $\mathcal{L}$ to be of order $r^{1/2}$ and keeping terms in the expansion to order $1/r$ gives

$$\frac{\mathcal{E}}{\mu}\geq 1-\frac{M-qQ/\mu}{r}+\frac{\mathcal{L}^2}{2\mu^2r^2}+..., \tag{30}$$

which leads to stable Keplerian-like orbits (the only difference is the magnitude of the central force, which is suppressed by electrostatic repulsion).

So long as $Q<M$, Eq. (29) gives a minimum energy $\mathcal{E}_{\rm min}(r)$ that is well-behaved at all $r>r_+$.

C. Neutral particles

The case of a neutral particle is not so different from the Schwarzschild case; the only difference is the additional Q^2/r^2. For example, the circular orbit of angular momentum $\mathcal{L}$ occurs at the minimum of

$$V_{\rm eff}(r)=\left(1-2\frac{M}{r}+\frac{Q^2}{r^2}\right)\left(1+\frac{\mathcal{L}^2}{\mu^2r^2}\right). \tag{31}$$

The ususal features such as a sequence of stable orbits, and ISCO, and then a sequence of unstable orbits occur in this case as well. [You will find them on the homework.]

D. Negatively charged particles

A negative particle should be attracted to a positive black hole; we might even expect the attraction to be stronger than the usual gravitational attraction. In fact, a negatively charged particle can have negative energy if it is close enough to the hole:

$$\lim_{r\to r_+}\mathcal{E}_{\rm min}(r)=\frac{qQ}{r_+}<0. \tag{32}$$

The deposition of such a particle can reduce the hole's gravitational mass! Since $\mathcal{E}_{\rm min}(r=\infty)=\mu$, it is not possible for a small particle to be dropped in from ∞ onto such an orbit, but a spacecraft near a black hole could in principle deploy such a particle on a negative energy orbit.

You might wonder now if we can make a charged black hole vanish ($M\to 0$) by dropping in opposite-sign test particles. This is not the case. To see this, we note that a negative particle dropped into the hole has $\mathcal{E}>qQ/r_+$ so that the change in the hole's mass is bounded from below:

$$\delta M>\frac{Q}{r_+}\delta Q. \tag{33}$$

If we consider the surface area of the black hole,

$$S=4\pi r_+^2, \tag{34}$$

then we see that

$$\begin{aligned}\delta S &= 8\pi r_+ \delta r_+ \\ &= 8\pi r_+ \left(\delta M + \frac{M\delta M - Q\delta Q}{\sqrt{M^2 - Q^2}}\right) \\ &> 8\pi r_+ \left(\frac{1}{r_+} + \frac{M/r_+ - 1}{\sqrt{M^2 - Q^2}}\right) Q\delta Q \\ &= 8\pi \left(1 + \frac{M - r_+}{\sqrt{M^2 - Q^2}}\right) Q\delta Q = 0. \end{aligned} \tag{35}$$

We conclude that by dropping in particles of negative charge, we may reduce the hole's mass, but **the surface area of the hole must increase** (or stay the same if we do nothing). This is a special case of a more general rule that we will prove, the *area theorem*.

E. Positively charged particles

A positively charged particle is repelled from a black hole. Nevertheless, it could be pushed in, and so we may ask: is it possible to push charge into a black hole to reach $Q = M$? If we drop in a charge $q > 0$, then in order to reach the horizon it must have $\mathcal{E} > qQ/r_+$. Then again

$$\delta M > \frac{Q}{r_+}\delta Q. \tag{36}$$

We then find that if we define $\chi = Q/M$:

$$\begin{aligned}\delta\chi &= \frac{\delta Q}{M} - \frac{\delta M}{M^2} \\ &> \frac{\delta Q}{M} - \frac{Q}{M^2 r_+}\delta Q \\ &= \left(1 - \frac{Q}{M r_+}\right)\frac{q}{M} \\ &= \left(1 - \frac{\chi}{1 + \sqrt{1 - \chi^2}}\right)\frac{q}{M}. \end{aligned} \tag{37}$$

The object in parentheses goes to zero as $\chi \to 1^-$. So as we push a black hole closer to $Q = M$, adding more charge adds more mass, so that the ratio χ asymptotes to 1: an infinite number of steps is required to reach $Q = M$.[1]

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[1] But it turns out that a finite amount of charge and mass are involved.