Lecture XXIV: External appearance of a black hole

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(Dated: February 15, 2012)

I. OVERVIEW

We now consider the problem of the appearance of a black hole formed by a collapsing star. Our interest today lies entirely with the appearance of the star long after its collapse, or alternatively of the radiation emitted as the star crosses the Schwarzschild radius. We first show that in geometric optics, the surface of the doomed star remains forever visible with exponentially decreasing brightness and increasing redshift. At some point, wave optics becomes important, and so we then treat that case as well.

Reading:

- MTW Ch. 31.
- This is not really reading material for this course, but the moving mirror example is from §4.4 of Birrell & Davies, *Quantum fields in curved spacetime*.

II. GEOMETRIC OPTICS TREATMENT

We begin with the Schwarzschild metric, in its new form:

$$ds^2 = \left(1 - 2\frac{M}{r}\right)(-dt^2 + dr_\star^2) + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{1}$$

with

$$r_\star = r + 2M \ln \frac{r - 2M}{2M}. \tag{2}$$

Of particular interest is the regime where r is slightly larger than $2M$: then $r = 2M(1+\epsilon)$, in which case

$$r_\star \approx 2M + 2M \ln \epsilon \tag{3}$$

and so

$$\epsilon \approx e^{r_\star/(2M)-1}. \tag{4}$$

For the case of a star whose "surface" is a shell in free-fall from infinity, with $u_\theta = u_\phi = 0$, $u_t = -1$, the normalization condition says that

$$\frac{-(u_t)^2 + (u_{r_\star})^2}{1 - 2M/r} = -1, \tag{5}$$

so that

$$u_{r_\star} = -\sqrt{\frac{2M}{r}} \tag{6}$$

and

$$\frac{dt}{dr_\star} = -\sqrt{\frac{r}{2M}}. \tag{7}$$

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If we're at small ϵ then:

$$-\frac{dt}{dr_\star} \approx 1 + \frac{1}{2}\epsilon \approx 1 + \frac{1}{2}e^{r_\star/(2M)-1}. \tag{8}$$

Now if we follow the light rays emitted from the surface of the star to an observer at ∞, what is of interest to us is the relation between proper time τ at the star and the outgoing null coordinate $\tilde{U} = t - r_\star$ (radially outgoing rays travel along curves of constant $\tilde{U}$). This is

$$\frac{d\tilde{U}}{d\tau} = u^t - u^{r_\star} = \left(1 - 2\frac{M}{r}\right)^{-1}(-u_t - u_{r_\star}) = \left(1 - 2\frac{M}{r}\right)^{-1}\left(1 + \sqrt{\frac{2M}{r}}\right) \approx 2\left(1 - 2\frac{M}{r}\right)^{-1} \approx \frac{2}{\epsilon}. \tag{9}$$

Thus the apparent redshift of the star is

$$1 + z \approx \frac{\epsilon}{2}, \tag{10}$$

which diverges as the star approaches the Schwarzschild radius.

Of direct observational interest however is not the redshift as a function of ϵ (not directly observable) but as a function of $\tilde{U}$ (which with some offset represents the proper time of a distant observer stationary relative to the hole – to a first approximation, us). To find this relation, take the trajectory of the star, Eq. (8), and transform to $d\tilde{U}/dr_\star$:

$$\frac{d\tilde{U}}{dr_\star} = \frac{dt}{dr_\star} - 1 \approx -2. \tag{11}$$

Integration gives

$$\tilde{U} \approx C + 4M - 2r_\star \approx C - 4M\ln\epsilon, \tag{12}$$

where C is an integration constant; alternatively $\epsilon \approx e^{(C-\tilde{U})/(4M)}$. It follows that

$$1 + z \approx \frac{2}{\epsilon} \propto e^{\tilde{U}/(4M)}. \tag{13}$$

It thus follows that **the redshift of the star's surface increases exponentially with time.** The rate of such increase is one e-fold every $\Delta t = 4M$.

For the formation of a $10M_\odot$ black hole, M in time units is $50\,\mu$s, so $4M = 200\,\mu$s. If the star were isolated and had a surface temperature at collapse of 10^{11} K (actually it is emitting neutrinos and is still surrounded by matter, but this does not change the essential point), then in only $\ln(10^{11}/3) = 24$ e-folds – just 1.2 ms – the star fades to the temperature of the CMB blackbody. So once the remnants of the star fall into the black hole, the black hole is really black (hence its name) and can only make its appearance known through gravity. It only "shines" if material is available to be accreted and emit radiation before plunging down the hole.

III. WAVE OPTICS TREATMENT

In the example above, the exponentially decreasing temperature of the star eventually falls to the point where the emitted radiation has a wavelength of order the size of the black hole. For a $10M_\odot$ black hole, at this point the frequency of emitted radiation is $\sim M^{-1} \sim 20$ kHz and hence not detectable (try even making a blackbody at this temperature in the lab, let alone seeing it!). But the analysis of this situation has profound implications for the ultimate fate of the hole, and perhaps for the interface between gravity and quantum mechanics.

We will consider the following simplified assumptions:

- The collapse is spherically symmetric and the surface layers have unit specific energy, i.e. are in free-fall from essentially infinite distance.
- The star's surface is a perfect conductor. (This is a boundary condition that turns out not to affect the final answer, but this is the simplest way to get there. It turns out that this is the opposite of a blackbody – a perfect conductor emits no light of its own – so this is not really consistent with the above analysis. Again it does not affect the essential aspect of the problem, which is that there is a first light ray that does not escape, and those near it are exponentially delayed.)
- The electromagnetic field can be treated as a scalar. (This is not really true, but the treatment of a "spin 1" vector particle has ancillary complications and index mechanics not in the "spin 0" scalar case, for no intuitive gain.)

A. Scalar fields in Schwarzschild spacetime

A massless scalar field in GR is describable by the equation:

$$\nabla_\alpha \nabla^\alpha \chi = 0, \tag{14}$$

which is the simplest possible wave equation. Such a scalar has a stress-energy tensor

$$T_{\mu\nu} = -(\nabla_\mu \chi)(\nabla_\nu \chi) - \frac{1}{2} g_{\mu\nu} (\nabla_\alpha \chi)(\nabla^\alpha \chi). \tag{15}$$

[It is straightforward to show that the wave equation gives ${T^{\mu\nu}}_{;\nu} = 0$.]

The scalar wave equation can be written as

$$\nabla_\alpha (g^{\alpha\beta} \partial_\beta \chi) = 0 \tag{16}$$

or

$$\partial_\alpha (g^{\alpha\beta} \partial_\beta \chi) + \Gamma^\alpha_{\alpha\gamma} g^{\gamma\beta} \partial_\beta \chi = 0. \tag{17}$$

Now we find that in $(t, r_\star, \theta, \phi)$ coordinates:

$$\Gamma^\alpha_{\alpha t} = \Gamma^\alpha_{\alpha\phi} = 0, \quad \Gamma^\alpha_{\alpha\theta} = \cot\theta, \quad \text{and} \quad \Gamma^\alpha_{\alpha r} = \frac{d}{dr_\star} \ln \left[r^2 \left(1 - \frac{2M}{r} \right) \right]. \tag{18}$$

The wave equation is thus

$$-\partial_t \left(\frac{1}{1 - 2M/r} \partial_t \chi \right) + \partial_{r_\star} \left(\frac{1}{1 - 2M/r} \partial_{r_\star} \chi \right) + \Gamma^\alpha_{\alpha r} \partial_{r_\star} \chi + \frac{1}{r^2} (\partial^2_\theta \chi + \cot\theta \, \partial_\theta \chi + \csc^2\theta \, \partial^2_\phi \chi) = 0. \tag{19}$$

Extracting the $1/(1 - 2M/r)$ from the partial derivative in $r_\star$ gives

$$-\partial_t \left(\frac{1}{1 - 2M/r} \partial_t \chi \right) + \frac{1}{1 - 2M/r} \partial^2_{r_\star} \chi + \left[\Gamma^\alpha_{\alpha r} - \frac{d}{dr_\star} \ln \left(1 - \frac{2M}{r} \right) \right] \partial_{r_\star} \chi + \frac{1}{r^2} (\partial^2_\theta \chi + \cot\theta \, \partial_\theta \chi + \csc^2\theta \, \partial^2_\phi \chi) = 0 \tag{20}$$

or

$$\frac{1}{1 - 2M/r} (-\partial^2_t \chi + \partial^2_{r_\star} \chi) + \frac{2(r - 2M)}{r^2} \partial_{r_\star} \chi + \frac{1}{r^2} (\partial^2_\theta \chi + \cot\theta \, \partial_\theta \chi + \csc^2\theta \, \partial^2_\phi \chi) = 0. \tag{21}$$

This is a standard separable wave equation in the angular coordinates: we may write

$$\chi(t, r_\star, \theta, \phi) = \sum_{\ell m} \chi_{\ell m}(t, r_\star) Y_{\ell m}(\theta, \phi), \tag{22}$$

where each multipole moment satisfies the equation

$$-\partial^2_t \chi_{\ell m} + \partial^2_{r_\star} \chi_{\ell m} + \frac{2(r - 2M)^2}{r^3} \partial_{r_\star} \chi_{\ell m} - \ell(\ell + 1) \frac{r - 2M}{r^3} \chi_{\ell m} = 0. \tag{23}$$

Equation (23) is a time-independent wave equation. At $r_\star \to -\infty$, it reduces to the standard 1+1 dimensional hyperbolic wave equation,

$$-\partial^2_t \chi_{\ell m} + \partial^2_{r_\star} \chi_{\ell m} \approx 0. \tag{24}$$

[At $r_\star \to \infty$, the standard spherical term $(2/r) \, \partial_{r_\star} \chi_{\ell m}$ enters, which causes emitted waves to have an amplitude that declines as $\propto 1/r$.] At $|r_\star|$ of order M or less (with factors that depend on the multipole ℓ), this equation has a complicated barrier structure. Its behavior will be that of any standard barrier penetration problem; a scalar wave incident on the black hole from afar (i.e. from $r_\star = \infty$) has some amplitude to be transmitted (absorbed into the hole) or reflected. Similarly, a wave launched outward from the hole (i.e. from $r_\star = -\infty$) may be transmitted (escape) or reflected (swallowed back by the hole).

On a time-independent background such as Schwarzschild, there is a conserved total energy of the wave,

$$\mathcal{E} = \int T_{\mu\nu}\xi^\mu n^\nu \, d^3V, \tag{25}$$

where d^3V is the 3-volume element, $\boldsymbol{n}$ is the surface normal, and $\boldsymbol{\xi}$ is the timelike Killing field. Using $\xi^\mu = (1,0,0,0)$ in the $(t,r_\star,\theta,\phi)$ coordinate system, $n^\nu = ((1-2M/r)^{-1/2},0,0,0)$, and a 3-volume element $r^2(1-2M/r)^{1/2}\sin\theta\, dr_\star\, d\theta\, d\phi$, and recalling that the multipole moments are independent (cross-terms don't contribute since the $Y_{\ell m}$ are orthogonal), we find that

$$\mathcal{E} = \sum_{\ell m}\int \left[\frac{1}{2}|\partial_t\chi_{\ell m}|^2 + \frac{1}{2}|\partial_{r_\star}\chi_{\ell m}|^2 + \frac{\ell(\ell+1)(r-2M)}{2r^3}|\chi_{\ell m}|^2\right] r^2\, dr_\star. \tag{26}$$

Again that this is conserved is easily verified.

Energy radiated from the collapsing black hole has a certain amount of energy per scalar wave mode, $N\sigma$ where σ is the frequency (and N is the action per wave – i.e. $\hbar$ times the mean occupation number in quantum language). Then the total power radiated from the hole is

$$P = \int_0^\infty \frac{d\sigma}{2\pi}\,\sigma \sum_{\ell=0}^{\infty}(2\ell+1)N_\ell(\sigma)\mathbb{T}_\ell(\sigma), \tag{27}$$

where the integral and sum count modes, and $\mathbb{T}_\ell(\sigma)$ is the power transmission coefficient. The problem now is to determine (i) the transmission coefficient, and (ii) the mode action escaping from the collapsing star. We'll focus on the latter as it is more physically interesting (the former will be of order unity for the modes of interest).

B. The falling mirror

Now let us consider the star just before it falls into the Schwarzschild radius. The surface is a "mirror" – for a scalar that means a Dirichlet boundary condition,

$$\chi(\text{surface}) = 0. \tag{28}$$

Since this boundary condition holds $\forall\ell m$, we will drop the clumsy subscripts. The solution to Eq. (24) is well-known to be a sum of arbitrary inward- and outward-going waves:

$$\chi(t,r_\star) = \chi_+(\tilde{U}) + \chi_-(\tilde{V}), \tag{29}$$

where again $\tilde{U} = t - r_\star$ and $\tilde{V} = t + r_\star$. The boundary condition tells us that

$$\chi_+(\tilde{U}) = -\chi_-(\tilde{V}) \quad \text{at boundary.} \tag{30}$$

the ingoing radiation is χ_-, and the outgoing radiation is χ_+. Thus we can solve for the latter in terms of the former, if we know the trajectory of the falling mirror through $(\tilde{U},\tilde{V})$-space.

Following Eq. (8), the boundary is falling inward according to

$$-\frac{dt}{dr_\star} \approx 1 + \frac{1}{2}e^{r_\star/(2M)-1}. \tag{31}$$

Thus

$$\frac{d\tilde{V}}{dr_\star} = 1 + \frac{dt}{dr_\star} = -\frac{1}{2}e^{r_\star/(2M)-1}, \tag{32}$$

so

$$\tilde{V} = B - Me^{r_\star/(2M)-1} \tag{33}$$

(for some B.) From Eq. (12), we have $\tilde{U} \approx C + 4M - 2r_\star$ and hence

$$\frac{r_\star}{2}M = -\frac{\tilde{U}-C}{4M} + 1, \tag{34}$$

and hence

$$\tilde{V} = B - Me^{(\tilde{U}-C)/(4M)}. \tag{35}$$

This relation, combined with Eq. (30), tells us the relation between ingoing and outgoing waves. We see that the wave coming out of the star is simply the ingoing wave, inverted and stretched. Only the wave incident at $\tilde{V} < B$ is returned – the very tail is stretched by a divergent amount.

Now if the ingoing wave were zero, as in the classical vacuum, the outgoing wave would be zero and this whole situation would be of no interest. But in reality, we know that the vacuum is full of quantum fluctuations: an ingoing mode of frequency σ has a mean energy per mode of $\frac{1}{2}\hbar\sigma$. Since half of this energy is potential and half kinetic, we have a contribution to the variance of χ:

$$\mathrm{Var}\,\chi_-(\tilde{V}) = \int_{\sigma_{\min}}^{\sigma_{\max}} (2M)^{-2}\frac{\hbar}{2\sigma}\frac{d\sigma}{2\pi}, \tag{36}$$

where $\int d\sigma/2\pi$ is the number of modes per unit $r_\star$ (for a given ℓm); and $(2M)^{-2}\hbar/(2\sigma)$ is the contribution of each mode to $\int \chi^2\,dr_\star$. It turns out that the variance is not enough for us, however – we need the correlation function (or covariance between the field at two different times with some separation $\Delta\tilde{V}$), which picks up an additional factor of $\cos\sigma\Delta\tilde{V}$ due to the dephasing of higher frequency modes. That is, we want the correlation function:

$$\xi(\Delta\tilde{V}) \equiv \langle\chi_-(\tilde{V})\chi_-(\tilde{V}+\Delta\tilde{V})\rangle = \int_0^\infty \frac{d\,\mathrm{Var}\,\chi_-(\tilde{V})}{d\sigma}\cos(\sigma\Delta\tilde{V})\,d\sigma \;\leftrightarrow\; \frac{d\,\mathrm{Var}\,\chi_-(\tilde{V})}{d\sigma} = \frac{2}{\pi}\int_0^\infty \xi(\Delta\tilde{V})\,\cos(\sigma\Delta\tilde{V})\,d\Delta\tilde{V}. \tag{37}$$

We then find:

$$\langle\chi_-(\tilde{V})\chi_-(\tilde{V}+\Delta\tilde{V})\rangle = \frac{1}{(2M)^2}\int_{\sigma_{\min}}^{\sigma_{\max}} \frac{\hbar}{2\sigma}\cos\sigma\Delta\tilde{V}\,\frac{d\sigma}{2\pi}. \tag{38}$$

We would like to take the limit as $\sigma_{\min} \to 0$, $\sigma_{\max} \to \infty$. There is no problem with the latter limit. The low-frequency limit however has a divergence, associated with large fluctuations in the field of very long wavelength. Clearly at some sufficiently low frequency $\sigma \sim |r_\star|^{-1}$ the machinery we've used breaks down. But we don't need to worry about the precise nature of the cutoff for our purposes – rather, we define $x = \sigma|\Delta\tilde{V}|$, and then

$$\langle\chi_-(\tilde{V})\chi_-(\tilde{V}+\Delta\tilde{V})\rangle = \frac{\hbar}{16\pi M^2}\int_{\sigma_{\min}|\Delta\tilde{V}|}^{\infty} x^{-1}\cos x\,dx. \tag{39}$$

So long as $\sigma \gg \sigma_{\min}$, the lower limit is at $x \ll 1$ where the integrand is $1/x$. Therefore we should be able to approximate the integral as a piece from $x < x_0 \ll 1$ (which is $-\ln x$) and some piece from $x > x_0$ (which is a constant A'). Thus:

$$\langle\chi_-(\tilde{V})\chi_-(\tilde{V}+\Delta\tilde{V})\rangle = \frac{\hbar}{16\pi M^2}(A' - \ln\sigma_{\min}|\Delta\tilde{V}|) = -\frac{\hbar}{16\pi M^2}\ln\frac{|\Delta\tilde{V}|}{A}, \tag{40}$$

where A is some divergent constant with units of time.

It is easy then to compute the expectation values of products of the outgoing field at two times $\tilde{U}_1$ and $\tilde{U}_2$. This is

$$\langle\chi_+(\tilde{U}_1)\chi_+(\tilde{U}_2)\rangle = -\frac{\hbar}{16\pi M^2}\ln\left[\frac{Me^{-C/(4M)}}{A}|e^{\tilde{U}_1/(4M)} - e^{\tilde{U}_2/(4M)}|\right]. \tag{41}$$

Now our interest is to find the contribution to the variance of χ_+ from waves of frequency σ. This is

$$\frac{d\,\mathrm{Var}\,\chi_+(\tilde{U})}{d\sigma} = \frac{2}{\pi}\int_0^\infty \langle\chi_+(\tilde{U}-\frac{1}{2}\varsigma)\chi_+(\tilde{U}+\frac{1}{2}\varsigma)\rangle\cos(\sigma\varsigma)\,d\varsigma. \tag{42}$$

[This is just an inverse Fourier transform – the inverse of Eq. (37).] After subtracting the constant off the Fourier transform, which is irrelevant except for DC modes, we get

$$\frac{d\,\mathrm{Var}\,\chi_+(\tilde{U})}{d\sigma} = \frac{\hbar}{8\pi^2M^2}\int_0^\infty \ln|e^{(\tilde{U}-\varsigma/2)/(4M)} - e^{(\tilde{U}+\varsigma/2)/(4M)}|\,\cos(\sigma\varsigma)\,d\varsigma. \tag{43}$$

Factoring out $e^{\tilde{U}/(4M)}$ gives another irrelevant constant, and we are left with

$$\frac{d\,\mathrm{Var}\,\chi_+(\tilde{U})}{d\sigma} = \frac{\hbar}{8\pi^2 M^2}\int_0^\infty \ln\left|\sinh\frac{\varsigma}{8M}\right|\,\cos(\sigma\varsigma)\,d\varsigma. \tag{44}$$

This is divergent at small ς. The divergence can be cured if we note that this variance contains both vacuum fluctuations, which contribute $-\hbar/(16\pi M^2)\ln\varsigma$ to the correlation function, and everything else. If we just count the "everything else" we get (again with an additive constant that only contributes to DC):

$$\left.\frac{d\,\mathrm{Var}\,\chi_+(\tilde{U})}{d\sigma}\right|_{\text{no vac.}} = \frac{\hbar}{8\pi^2 M^2}\int_0^\infty \ln\frac{\sinh[\varsigma/(8M)]}{\varsigma/(8M)}\,\cos(\sigma\varsigma)\,d\varsigma. \tag{45}$$

The integral is now well-behaved at $\varsigma = 0$. [It blows up at $\varsigma\to\infty$, but in an oscillatory way so that it can be regularized by inserting a factor of $e^{-\mu\varsigma}$ for infinitesimal μ, or equivalently by using complex exponentials and deforming the contour so that ς has a positive imaginary part as we will do below.]

Our final problem is to evaluate the integral in Eq. (45). This can be done by first taking the integral from $-\infty$ to ∞ and dividing by 2; also the cosine is the real part of a complex exponential:

$$\left.\frac{d\,\mathrm{Var}\,\chi_+(\tilde{U})}{d\sigma}\right|_{\text{no vac.}} = \frac{\hbar}{16\pi^2 M^2}\Re\int_{-\infty}^\infty \ln\frac{\sinh[\varsigma/(8M)]}{\varsigma/(8M)}\,e^{i\sigma\varsigma}\,d\varsigma. \tag{46}$$

We would like to do a contour integration and close the contour in the upper half-complex plane. However we can't do this right now because the ln has a branch cut. We cure this by taking $e^{i\sigma\varsigma} = (-i/\sigma)\partial_\varsigma(e^{i\sigma\varsigma})$ and performing an integration by parts to move the derivative onto the logarithm:

$$\begin{aligned}
\left.\frac{d\,\mathrm{Var}\,\chi_+(\tilde{U})}{d\sigma}\right|_{\text{no vac.}} &= \frac{\hbar}{16\pi^2 M^2}\Re\frac{-i}{\sigma}\int_{-\infty}^\infty \left(\frac{\sinh[\varsigma/(8M)]}{\varsigma/(8M)}\right)^{-1} \\
&\quad\times\frac{[\varsigma/(8M)][1/(8M)]\cosh[\varsigma/(8M)] - [1/(8M)]\sinh[\varsigma/(8M)]}{[\varsigma/(8M)]^2}\,e^{i\sigma\varsigma}\,d\varsigma \\
&= -\frac{\hbar}{16\pi^2 M^2\sigma}\Re i\int_{-\infty}^\infty \left\{\frac{1}{8M}\coth\frac{T}{8M} - \frac{1}{\varsigma}\right\}e^{i\sigma\varsigma}\,d\varsigma \\
&= -\frac{\hbar}{16\pi^2 M^2\sigma}\Re i\int_{-\infty}^\infty \left\{\frac{1}{8M}\frac{e^{\varsigma/(4M)}+1}{e^{\varsigma/(4M)}-1} - \frac{1}{\varsigma}\right\}e^{i\sigma\varsigma}\,d\varsigma.
\end{aligned} \tag{47}$$

Now the contour can be closed in the upper-half plane, but there are simple poles at the locations where $e^{\varsigma/(4M)} = 1$, i.e. at $\varsigma = 8\pi i n M$ for $n\in\mathbb{Z}\setminus\{0\}$ ($\varsigma = 0$ has a term that cancels the pole). The residues are obtained by the usual prescription of replacing the denominator with its derivative $e^{\varsigma/(4M)}/(4M) = 1/(4M)$, so we find

$$\mathrm{Res}_{\varsigma=8\pi i n M}\left\{\frac{1}{8M}\frac{e^{\varsigma/(4M)}+1}{e^{\varsigma/(4M)}-1} - \frac{1}{\varsigma}\right\}e^{i\sigma\varsigma} = e^{-8\pi n M\sigma}. \tag{48}$$

The integral is then $2\pi i$ times the sum of the residues, which leads to

$$\left.\frac{d\,\mathrm{Var}\,\chi_+(\tilde{U})}{d\sigma}\right|_{\text{no vac.}} = -\frac{\hbar}{16\pi^2 M^2\sigma}\Re - 2\pi\sum_{n=1}^\infty e^{-8\pi n M\sigma} = \frac{\hbar}{8\pi M^2\sigma(e^{8\pi M\sigma}-1)}. \tag{49}$$

Now we are interested in the action per mode, i.e. the energy per mode divided by σ. This is obtained from our usual formula for the contribution to the variance of χ:

$$\left.\frac{d\,\mathrm{Var}\,\chi_+(\tilde{U})}{d\sigma}\right|_{\text{no vac.}} d\sigma = \frac{1}{(2M)^2}\frac{N(\sigma)}{\sigma}\frac{d\sigma}{2\pi}. \tag{50}$$

Thus we see that

$$N(\sigma) = 8\pi M^2\hbar\sigma\,\left.\frac{d\,\mathrm{Var}\,\chi_+(\tilde{U})}{d\sigma}\right|_{\text{no vac.}} = \frac{\hbar}{e^{8\pi M\sigma}-1}. \tag{51}$$

This is the formula for blackbody radiation at a temperature of $T = 1/(8\pi M)$. Equation (27) then gives the total emitted power

$$P = \hbar \int_0^\infty \frac{d\sigma}{2\pi}\,\sigma \sum_{\ell=0}^{\infty}(2\ell+1)\frac{1}{e^{8\pi M\sigma}-1}\mathbb{T}_\ell(\sigma). \tag{52}$$

We thus see that **the black hole appears to radiate as if the inner boundary condition were a perfect blackbody at temperature** $1/(8\pi M)$. This effect is known as *Hawking radiation*.

IV. IMPLICATIONS OF HAWKING RADIATION

Let us first estimate the Hawking-radiation luminosity of a black hole. The transmission coefficient $T_\ell(\sigma)$ is of order unity for $\sigma \sim M$ and the lowest ℓs, so the integration will lead to

$$P = \frac{\hbar K}{M^2} = \frac{\hbar c^6 K}{G^2 M^2}, \tag{53}$$

where K is some factor of order unity. The actual value of K depends on the number and spin of the various massless particles. At least the photon (spin 1) and graviton (spin 2) are massless. Whether there are other such fields is an open question. It also turns out that "light" particles (neutrinos) can be produced according to the above calculation if their Compton wavelength is $\gg M$, i.e. for a black hole of sufficiently low mass.

The direct emission of Hawking radiation is astrophysically irrelevant. It is easily seen that

$$\frac{\hbar c^6}{G^2 M^2} = 3\times 10^{-16}\left(\frac{M_\odot}{M}\right)^2 \text{ erg/s}. \tag{54}$$

Detailed calculations taking account of the factors of 8π give a result about 3 orders of magnitude smaller.

However the Hawking radiation has a fundamental implication for black holes: they live a finite period of time! In particular, a black hole of mass M should emit its entire mass-energy in Hawking radiation in a time

$$t_{\rm HR} \sim 10^3 \frac{Mc^2}{P} \sim 10^3 \frac{G^2 M^3}{\hbar c^4} \sim 10^{73}\left(\frac{M}{M_\odot}\right)^3 \text{ s}. \tag{55}$$

This is many orders of magnitude longer than the age of the Universe – but it means that black holes are not forever!

There is also an implication for the range of possible masses of a black hole. The concept of a Schwarzschild-like black hole ceases to make sense if the hole lives for less than a light-crossing time, i.e. if

$$t_{\rm HR} \sim \frac{G^2 M^3}{\hbar c^4} < \frac{GM}{c^3} \tag{56}$$

or

$$M < M_{\rm Pl} = \sqrt{\frac{\hbar c}{G}} = 5\times 10^{-5}\text{ g}. \tag{57}$$

This is the *Planck mass*, and below it classical GR does not provide a description of a black hole. The ultimate fate of a black hole that radiates away its mass and reaches the Planck mass (does it completely evaporate, or leave behind a stable $M \sim M_{\rm Pl}$ relic?) is unknown.