# Lecture XXIII: The Schwarzschild black hole


Christopher M. Hirata
*Caltech M/C 350-17, Pasadena CA 91125, USA*[*]



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## I. OVERVIEW

Having considered the subject of relativistic stars, we now turn to the problem of spherically symmetric black holes. Our first step is to understand the global geometry of the Schwarzschild hole. We will then proceed to consider the formation and stability of such a hole.

Reading:

- MTW Ch. 31.

## II. THE BEHAVIOR AT $r = 2M$

We begin with the Schwarzschild metric:

$$ds^2 = -\left(1 - 2\frac{M}{r}\right) dt^2 + \frac{dr^2}{1 - 2M/r} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{1}$$

This describes the exterior ($r > R$) of a star of mass $M$. We will now proceed to consider the case of a metric where this exterior geometry continues to hold all the way down to arbitrarily small radii. This spacetime is called *Schwarzschild spacetime*. It is obviously well behaved for $r > 2M$.

What is less obvious about Schwarzschild is its behavior when $r = 2M$, where $g_{rr} \rightarrow \infty$ and $g_{tt} \rightarrow 0$. The system appears singular; but is this a real singularity, or is it an artifact of the coordinate system? We will eventually answer the latter by constructing a well-behaved coordinate system, but let us first consider the plight of the hapless observer $\mathcal{O}$ who dives into Schwarzschild. What does he/she experience?

### A. Radial coordinate of an infalling observer

For simplicity, let's assume that $\mathcal{O}$ has zero angular momentum, so travels along a curve with $(\theta, \phi)$ constant, and unit energy per unit mass – that is, $u_t = -1$. Then the shell condition $g^{\alpha\beta}u_\alpha u_\beta = -1$ tells us that

$$-\frac{(u_t)^2}{1 - 2M/r} + \left(1 - 2\frac{M}{r}\right)(u_r)^2 = -1, \tag{2}$$

or setting $u_t = -1$ and solving for $u_r$:

$$(u_r)^2 = \frac{-1 + 1/(1 - 2M/r)}{1 - 2M/r} = \frac{2M/r}{(1 - 2M/r)^2}, \tag{3}$$

so taking the inward-going route: $u_r = -\sqrt{2M/r}\,/(1 - 2M/r)$ and

$$\frac{dr}{d\tau} = u^r = -\sqrt{\frac{2M}{r}}. \tag{4}$$

Now $\mathcal{O}$'s radial coordinate as a function of proper time is thus obtained from

$$r^{1/2}\, dr = -\sqrt{2M}\, d\tau \tag{5}$$


[*]Electronic address: `chirata@tapir.caltech.edu`

or

$$r = \left(\frac{9M}{2}\right)^{1/3} (\tau_0 - \tau)^{2/3}, \tag{6}$$

where $\tau_0$ is an integration constant representing the time at which the observer would reach $r = 0$ (if they survive that long). Note that the observer reaches $r = 2M$ and even $r = 0$ in finite proper time!

### B. Riemann tensor

We ask at this point what curvature tensor is seen by $\mathcal{O}$. To answer this question, we compute the Riemann tensor, first in the orthonormal basis $\{\boldsymbol{e}_{\hat{t}}, \boldsymbol{e}_{\hat{r}}, \boldsymbol{e}_{\hat{\theta}}, \boldsymbol{e}_{\hat{\phi}}\}$, and then transform to $\mathcal{O}$'s rest frame. The nonzero Riemann tensor components are:

$$R_{\hat{t}\hat{r}\hat{t}\hat{r}} = -2\frac{M}{r^3}, \quad R_{\hat{t}\hat{\theta}\hat{t}\hat{\theta}} = R_{\hat{t}\hat{\phi}\hat{t}\hat{\phi}} = \frac{M}{r^3}, \quad R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = 2\frac{M}{r^3}, \quad \text{and} \quad R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}} = R_{\hat{r}\hat{\phi}\hat{r}\hat{\phi}} = -\frac{M}{r^3}. \tag{7}$$

However the observer has an inward 4-velocity that is some linear combination of $\boldsymbol{e}_{\hat{r}}$ and $\boldsymbol{e}_{\hat{t}}$:

$$\boldsymbol{e}_{\hat{\tau}} \equiv \boldsymbol{u} = \frac{\boldsymbol{e}_{\hat{t}} - v\boldsymbol{e}_{\hat{r}}}{\sqrt{1 - v^2}}, \tag{8}$$

and a corresponding spatial basis vector

$$\boldsymbol{e}_{\hat{\rho}} = \frac{\boldsymbol{e}_{\hat{r}} - v\boldsymbol{e}_{\hat{t}}}{\sqrt{1 - v^2}}. \tag{9}$$

Thus the observer sees $\{\boldsymbol{e}_{\hat{\rho}}, \boldsymbol{e}_{\hat{\theta}}, \boldsymbol{e}_{\hat{\phi}}\}$ as spatial basis vectors. A straightforward calculation gives the transformation

$$R_{\hat{\tau}\hat{\rho}\hat{\tau}\hat{\rho}} = -2\frac{M}{r^3}, \quad R_{\hat{\tau}\hat{\theta}\hat{\tau}\hat{\theta}} = R_{\hat{\tau}\hat{\phi}\hat{\tau}\hat{\phi}} = \frac{M}{r^3}, \quad R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = 2\frac{M}{r^3}, \quad \text{and} \quad R_{\hat{\rho}\hat{\theta}\hat{\rho}\hat{\theta}} = R_{\hat{\rho}\hat{\phi}\hat{\rho}\hat{\phi}} = -\frac{M}{r^3}. \tag{10}$$

This is perfectly well behaved at $r = 2M$. Therefore we conclude that an observer doing local measurements of nearby test particles will not see anything unusual happen as they cross $r = 2M$. However, when they reach $r = 0$ the curvature blows up. This represents a physical *singularity* of the system that cannot be removed by a choice of coordinates.

### C. How long does it take to reach $r = 2M$?

But $\mathcal{O}$ does not reach $r = 2M$ in finite coordinate time. This is because if we solve for $u^t$ we have:

$$u^t = \frac{1}{1 - 2M/r} \quad \rightarrow \quad \frac{dt}{dr} = \frac{u^t}{u^r} = -\frac{1/(1 - 2M/r)}{\sqrt{2M/r}} = -\frac{1}{(1 - 2M/r)\sqrt{2M/r}}. \tag{11}$$

The coordinate time for the observer's trajectory is then

$$t = -\int \frac{dr}{(1 - 2M/r)\sqrt{2M/r}}, \tag{12}$$

or setting $r = 2Mz^2$ and hence $dr = 4Mz\,dz$:

$$\begin{aligned} t &= -4M \int \frac{z\,dz}{(1 - z^{-2})z^{-1}} \\ &= -4M \int \frac{z^4\,dz}{z^2 - 1} \\ &= -4M \int \left( z^2 + 1 + \frac{1}{2(z-1)} - \frac{1}{2(z+1)} \right) dz \\ &= -4M \left( \frac{1}{3}z^3 + z + \frac{1}{2}\ln\frac{z-1}{z+1} \right) + t_0 \\ &= -4M \left( \frac{r}{6M}\sqrt{\frac{r}{2M}} + \sqrt{\frac{r}{2M}} + \frac{1}{2}\ln\frac{\sqrt{r/2M} - 1}{\sqrt{r/2M} + 1} \right) + t_0. \end{aligned} \tag{13}$$

At large radii $r$, this is perfectly well-behaved. However as $r = 2M(1+\epsilon)$, with $\epsilon \to 0^+$, we have the limit that

$$t \to t_0 - 2M \ln \frac{\sqrt{1+\epsilon} - 1}{2} \approx t_0 + 2M \ln \frac{4}{\epsilon} \to +\infty. \tag{14}$$

Thus as far as the coordinate time is concerned – or as far as any external observer can see – $\mathcal{O}$ takes forever (literally) to reach $r = 2M$. As seen by $\mathcal{O}$ themselves, however, the amount of time required is finite.

To summarize: **the "singularity" at $r = 2M$ is an artifact of the coordinate system, and it can be reached by an observer who remains alive and well; but the singularity at $r = 0$ is real**. At $r = 0$, the known laws of physics must break down: basic operations such as derivatives or even defining a continuous function become impossible.

## III. KRUSKAL-SZEKERES COORDINATES

That there is no actual singularity at $r = 2M$ suggests that we might build a new coordinate system without such pathological behavior. This system will require no modification to $\theta$ or $\phi$, but will require new coordinates to replace $r$ and (surprisingly) $t$.

The basis of this new coordinate system is the set of null geodesics emanating from and disappearing into the central body. A radial null geodesic must have

$$-\left(1 - 2\frac{M}{r}\right)(u^t)^2 + \frac{(u^r)^2}{1 - 2M/r} = 0. \tag{15}$$

Then

$$\frac{dt}{dr} = \pm \frac{1}{1 - 2M/r}. \tag{16}$$

We define the *tortoise coordinate* $r_\star$ to be a rescaled version of $r$ for which radial null geodesics are 45° lines in the $(t, r_\star)$ plane. That is, we want $dt/dr_\star = \pm 1$. Thus we want to choose $r_\star$ to be given by

$$r_\star = \int \frac{dr}{1 - 2M/r} = \int \left(1 + \frac{2M}{r - 2M}\right) dr = r + 2M \ln \frac{r - 2M}{2M} \tag{17}$$

(we may simply choose the integration constant this way as we use this equation to define $r_\star$).

Mathematically the new coordinate $r_\star$ is simply a change of variable from $r$, but it has a different range of applicability: the regime $2M < r < \infty$ is mapped into $-\infty < r_\star < \infty$. There is also formally a function $r(r_\star)$, but since solving Eq. (17) for $r$ does not have an analytic solution in terms of simple functions we won't write it down. Taking $r_\star$ in place of $r$, then, we have

$$ds^2 = \left(1 - 2\frac{M}{r}\right)(-dt^2 + dr_\star^2) + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{18}$$

Here $r$ is a function of $r_\star$. The "normal" part of the spacetime, $r > 2M$, is now mapped into the plane $(t, r_\star) \in \mathbb{R}^2$.

The next step is to define a rotated coordinate system, where instead of specifying a point by $(t, r_\star)$ we specify which outgoing null ray and which ingoing null ray intersect there. This is equivalent to rotating 45°:

$$\tilde{V} - \tilde{U} = 2r_\star \quad \text{and} \quad \tilde{V} + \tilde{U} = 2t. \tag{19}$$

Again the "normal" part of the spacetime is now in $(\tilde{U}, \tilde{V}) \in \mathbb{R}^2$, and noting that $-dt^2 + dr_\star^2 = -d\tilde{U}\, d\tilde{V}$ we find:

$$ds^2 = -\left(1 - 2\frac{M}{r}\right) d\tilde{U}\, d\tilde{V} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{20}$$

Here again $r$ is a function: $r(\tilde{U}, \tilde{V})$.

In the $(\tilde{U}, \tilde{V})$ system, it is readily seen that objects moving "forward in time" (i.e. on timelike trajectories with $dt/d\tau > 0$) must have $d\tilde{U}/d\tau > 0$ and $d\tilde{V}/d\tau > 0$. Thus they must be moving to the upper-right (first quadrant).

Unfortunately, Eq. (20) still does not eliminate the troublesome coordinate singularity at $r = 2M$. This can be done by rescaling $\tilde{U}$ and $\tilde{V}$ to new variables $\tilde{u}(\tilde{U})$ and $\tilde{v}(\tilde{V})$ defined monotonically, which preserves the statement that observers must move to the upper-right. Under such a rescaling, we have

$$g_{\tilde{u}\tilde{v}} = g_{\tilde{U}\tilde{V}} \frac{\partial \tilde{U}}{\partial \tilde{u}} \frac{\partial \tilde{V}}{\partial \tilde{v}} \equiv \frac{g_{\tilde{U}\tilde{V}}}{f}, \tag{21}$$

so in order to eliminate the zero-crossing at $r - 2M$ in $g_{\tilde{U}\tilde{V}}$, we want $f \equiv (\partial\tilde{u}/\partial\tilde{U})(\partial\tilde{v}/\partial\tilde{V})$ to be of the form $r - 2M$ times an analytic nonzero function. We note that from Eq. (17)

$$e^{r_\star/(2M)} = \left(\frac{r}{2M} - 1\right) e^{r/(2M)}, \tag{22}$$

so having a factor of $e^{r_\star/(2M)}$ in $f$ would satisfy our requirements. In fact, since

$$e^{r_\star/(2M)} = e^{(\tilde{V}-\tilde{U})/(4M)} = e^{\tilde{V}/(4M)} e^{-\tilde{U}/(4M)}, \tag{23}$$

our problem separates nicely: we could have $f \propto e^{r_\star/(2M)}$ by choosing

$$\frac{\partial \tilde{u}}{\partial \tilde{U}} \propto e^{-\tilde{U}/(4M)} \quad \text{and} \quad \frac{\partial \tilde{v}}{\partial \tilde{V}} \propto e^{\tilde{V}/(4M)}. \tag{24}$$

(The constants of integration are irrelevant.) Thus we choose

$$\tilde{u} = -e^{-\tilde{U}/(4M)} \quad \text{and} \quad \tilde{v} = e^{\tilde{V}/(4M)}. \tag{25}$$

[We inserted the $-$ sign in front of $\tilde{u}$ to ensure that $\partial\tilde{u}/\partial\tilde{U} > 0$ and hence we don't flip the spacetime coordinates.] Now $f = e^{r_\star/(2M)}/(4M)^2$, so that

$$g_{\tilde{u}\tilde{v}} = -\frac{1 - 2M/r}{e^{r_\star/(2M)}/(4M)^2} = -e^{-r/(2M)} \frac{(4M)^2(1 - 2M/r)}{(r/2M - 1)} = -\frac{32M^3}{r} e^{-r/(2M)}. \tag{26}$$

Thus:

$$ds^2 = -\frac{32M^3}{r} e^{-r/(2M)} d\tilde{u}\, d\tilde{v} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{27}$$

In Eq. (27), $r$ is a function $r(\tilde{u}, \tilde{v})$. This function itself has no simple analytic form. The inverse transformation, however, is easy to express:

$$\tilde{u} = -e^{-(t-r_\star)/(4M)} = -\sqrt{\frac{r-2M}{2M}} e^{r/(4M)} e^{-t/(4M)} \quad \text{and} \quad \tilde{v} = e^{(t+r_\star)/(4M)} = \sqrt{\frac{r-2M}{2M}} e^{r/(4M)} e^{t/(4M)}. \tag{28}$$

This is the *Kruskal-Szekeres coordinate system.*

[Note: MTW does a further rotation of this system by 45° to $(u, v, \theta, \phi)$; I won't do that. I trust that you can think of particles having to increase both $\tilde{u}$ and $\tilde{v}$ along their world lines. I will draw the diagram at a 45° angle if you like.]

The relationship with $r$ is key to understanding the Kruskal-Szekeres coordinate system. We note that

$$e^{r_\star/(2M)} = -\tilde{u}\tilde{v} \quad \text{and} \quad t = 2M \ln\left(-\frac{\tilde{v}}{\tilde{u}}\right), \tag{29}$$

so the normal region with $r > 2M$ and $-\infty < t < \infty$ occupies the 2nd quadrant, $\tilde{u} < 0$ and $\tilde{v} > 0$. The entire quadrant is legal – $e^{r_\star/(2M)}$ can have any positive value – and while all particles move to the upper-right, curves of constant $r$ are described by hyperbolas with $\tilde{u}\tilde{v} = $constant. Furthermore, we can see that the straight lines through the "origin" are curves of constant $t$.

[**Warning**: MTW, like many references, refers to the 1st, 2nd, 3rd, and 4th quadrants in the $(\tilde{u}, \tilde{v})$ plane as, respectively, regions II, I, IV, and III. I find this very confusing, but I suppose it makes sense if we live in region I.]

The location where the previous coordinate system was ill-behaved was at $r = 2M$ or $r_\star = -\infty$, which is now described by the "crosshairs" $\tilde{u}\tilde{v} = 0$. A particle approaching this from the 2nd quadrant (e.g. the ill-fated $\mathcal{O}$ from the previous section!) reaches this surface at $t = \infty$. Since the observer must move to the upper-right, he/she is now forbidden from crossing back into the 2nd quadrant, i.e. into the "normal" portion of the Universe. Such a zone, compact as viewed from the outside (it is restricted to small $r$) but from which there is no escape, is called a *black hole*. Its edge, $r = 2M$ in this case, is called the *event horizon*.

So what happens inside a black hole? Well, this is the 1st quadrant now. The radius coordinate $r$ does not do anything special. Indeed, we can see from Eq. (17) that

$$-\tilde{u}\tilde{v} = e^{r_\star/2M} = -\frac{r-2M}{2M}e^{r/(2M)}. \tag{30}$$

So even though $r_\star$ is now imaginary (the logarithm of a negative number), the function $(r-2M)e^{r/M}/(2M)$ can go negative for $r < 2M$. However, $r$ reaches zero at $\tilde{u}\tilde{v} = 1$. This is the final singularity. Note that it cannot be avoided: the observer is moving to the upper-right, and once they have crossed the event horizon their doom is sealed.

The spacetime admits other regions as well. If we flip the sign of both $\tilde{u}$ and $\tilde{v}$, then both $\tilde{u}\tilde{v}$ (and hence $r$) and $\tilde{v}/\tilde{u}$ (and hence $t$) retain their original values. Thus, if one takes seriously the analytic continuation of the Schwarzschild metric (which one should not if the black hole was formed by a collapsing star!) then it contains:

- **1st quadrant** [II]: The interior of the black hole, $r < 2M$. Particles may move either direction in $t$, but always to smaller $r$. This quadrant is cut off at the *final singularity* at $\tilde{u}\tilde{v} = 1$.
- **2nd quadrant** [I]: The normal exterior of the hole. Particles may move either direction in $r$, but always forward in $t$. Particles that reach the event horizon at $\tilde{u} = 0$ cross into the hole and can no longer send signals to the 2nd quadrant.
- **3rd quadrant** [IV]: The past interior of the black hole. This is bounded by the *past singularity* at $\tilde{u}\tilde{v} = 1$. Particles in this region may move either direction in $t$, but always to larger $r$. They may choose to reach either the 2nd or 4th quadrants; but entering one or the other is their destiny.
- **4th quadrant** [III]: A mirror universe, asymptotically flat and identical to the 2nd quadrant. Particles may move either direction in $r$, but always backward in $t$. Particles that reach the event horizon at $\tilde{v} = 0$ cross into the hole (1st quadrant) and are doomed.

Particles that cross from the 3rd to the 2nd quadrant emerge at $r = 2M$ and $t = -\infty$; this fountain of stuff from the infinite past is called a *white hole*; but we will see that these are not expected in nature (unless somehow present in the initial conditions). If one takes a spatial slice such as $\tilde{u} + \tilde{v} = 0$ through the metric, then one can see that the normal and mirror universes are connected via an *Einstein-Rosen bridge* with a "throat" of circumference $2\pi r(0,0) = 4\pi M$. Unfortunately, such a bridge cannot be crossed; anyone who tried would become trapped in the 1st quadrant and hit the singularity.

**But, we will see later that the physically relevant portions of the Schwarzschild spacetime are the subset of the 1st and 2nd quadrants exterior to a collapsing star**. Thus the event horizon and final singularity are actually realized, but (sadly) the mirror universe and the white hole are mere analytic continuations, and are as unphysical as the indefinite extrapolation of the arc of a basketball along an elliptical trajectory through Earth with one focus at the planet's center.

## IV. THE PENROSE DIAGRAM

The preceding machinery is sufficient to describe Schwarzschild geometry, but in studying more complicated black holes we need one more transformation to make each of the above regions finite in extent. This transformation is

$$\tilde{u} = \tan\xi \quad \text{and} \quad \tilde{v} = \tan\eta. \tag{31}$$

Then

$$ds^2 = -\frac{32M^3}{r}e^{-r/(2M)}\sec^2\xi\sec^2\eta\,d\xi\,d\eta + r^2(d\theta^2 + \sin^2\theta\,d\phi^2). \tag{32}$$

This maps the entire universe, including its analytic continuation, into a subset of $-\frac{\pi}{2} < \xi < \frac{\pi}{2}$, $-\frac{\pi}{2} < \eta < \frac{\pi}{2}$. The future singularity (in the 1st quadrant) is now $\xi + \eta = \frac{\pi}{2}$, and the past singularity (in the 3rd quadrant) is $\xi + \eta = -\frac{\pi}{2}$.

Such a picture of a spacetime, with each of its major regions shown as a finite open region and with the causal structure on display by having two of the coordinates defined by sets of null curves propagating in opposite directions, is called a *Penrose diagram*. The Penrose diagram is useful in a great many problems. Most importantly, it shows the causal structure of the spacetime, and allows us to talk about the edges of the diagram − in this case, for example, radiation escaping from the system to infinity goes to *future null infinity* ($-\frac{\pi}{2} < \xi < 0$, $\eta = \frac{\pi}{2}$), and *spatial infinity* is at $(-\frac{\pi}{2}, \frac{\pi}{2})$. We will see such diagrams again for rotating black holes, and in cosmology.