Lecture XXII: Radial pulsations and stability – implications

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I. OVERVIEW

We are now ready to investigate the stability against radial perturbations of stars. We will investigate both the case of purely Newtonian stars, as well as weakly relativistic stars in the critical case that $\Gamma_1 \approx \frac{4}{3}$. Note here that we consider only stability against radial perturbations – convective instabilities (as occur in the outer layers of the Sun!) are not described.

Reading:

- MTW Ch. 26.

II. REPRISAL

From the previous lecture, we recall that a star is stable against radial perturbations if the energy functional

$$\mathcal{E}[\zeta] \equiv \frac{\int_0^R (P\zeta'^2 - Q\zeta^2)\,dr}{\int_0^R W\zeta^2\,dr} \tag{1}$$

is positive for all perturbations $\zeta = r^2 e^{-\Phi_\circ}\xi$ obeying the boundary conditions

$$\lim_{r\rightarrow 0^+} \frac{\zeta}{r^3} = \text{finite} \quad \text{and} \quad \lim_{r\rightarrow R^-} (\Gamma_1 p\zeta') = 0. \tag{2}$$

Here the coefficients in the equation are

$$\begin{aligned} W &= r^{-2}(\rho+p)e^{3\Lambda-\Phi}, \\ P &= \Gamma_1 p r^{-2} e^{\Lambda+3\Phi}, \quad \text{and} \\ Q &= e^{\Lambda+3\Phi}\left[\frac{p'^2}{r^2(\rho+p)} + 4\frac{(\rho+p)(m+2\pi r^3 p)}{r^4(r-2m)}\right]. \end{aligned} \tag{3}$$

The eigenfrequencies σ are found from the equation

$$\sigma^2 W\zeta = -(P\zeta')' - Q\zeta. \tag{4}$$

In analogy to quantum-mechanical language, one could treat two possible functions ζ_1 and ζ_2 as having an "inner product"

$$\langle\zeta_1|\zeta_2\rangle = \int_0^R W\zeta_1^*\zeta_2\,dr \tag{5}$$

with a Hilbert space of functions satisfying the boundary condition. If one constructs the "Hamiltonian" defined by the relation

$$\langle\zeta_1|H|\zeta_2\rangle = \int_0^R (P\zeta_1'^*\zeta_2' - Q\zeta_1^*\zeta_2)\,dr \tag{6}$$

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which is explicitly Hermitian, then the Hamiltonian function is

$$H[\zeta] = -\frac{1}{W}(P\zeta')' - \frac{Q}{W}\zeta \tag{7}$$

and its eigenvalues are the σ^2 corresponding to normal modes. This allows us to use all of the machinery of quantum mechanics, including the variational principle.

We now examine several limiting cases.

III. NEWTONIAN STARS

The simplest case is that of a Newtonian star. In this case, we have $p' = -\rho m/r^2$ and the potentials are small, hence

$$W = \frac{\rho}{r^2}, \quad P = \Gamma_1\frac{p}{r^2}, \quad \text{and} \quad Q = -4\frac{p'}{r^3}. \tag{8}$$

Let us begin by considering a trial function with no nodes, which might make a reasonable approximation to the ground state (lowest mode): $\zeta = \epsilon r^3$, which corresponds to a homologous expansion or contraction. Such a trial gives

$$\mathcal{E}[\zeta] = \frac{\int_0^R (9\Gamma_1 pr^2 + 4p'r^3)\,dr}{\int_0^R \rho^2 r^4\,dr}. \tag{9}$$

Integrating by parts in the numerator gives

$$\mathcal{E}[\zeta] = \frac{16pr^3|_{r=0}^R + \int_0^R (9\Gamma_1 pr^2 - 12pr^2)\,dr}{\int_0^R \rho^2 r^4\,dr}. \tag{10}$$

The surface term vanishes (why?) and we are left with the conclusion that the homologous trial function has nonnegative energy if and only if

$$\int_0^R \left(\Gamma_1 - \frac{4}{3}\right) pr^2\,dr \ge 0. \tag{11}$$

Any Newtonian star not satisfying Eq. (11) is unstable! This immediately implies the instability of the sequence of stars intermediate between white dwarfs and neutron stars, for which most of the interior has a soft equation of state. [Since $M/R \ll 1$ for these objects, and $\Gamma_1 - \frac{4}{3}$ is of order unity, relativistic corrections cannot save these objects.]

Note that we have achieved this result without any explicit solution of the eigenvalue equation; we have not found the unstable mode. Such is the great power of the variational principle.

We have also identified $\frac{4}{3}$ as a critical value of the adiabatic exponent; this number determines the fate of stars.

A. Stars with $\Gamma_1 \ge \frac{4}{3}$

Let us next consider the case of a star with constant $\Gamma_1 = \frac{4}{3}$. What is to become of its perturbations? We note that the energy functional is

$$\mathcal{E}[\zeta] = \frac{\int_0^R (\Gamma_1 pr^{-2}\zeta'^2 + 4p'r^{-3}\zeta^2)\,dr}{\int_0^R \rho r^{-2}\zeta^2\,dr}. \tag{12}$$

Applying integration by parts to the numerator, we find that it is

$$\begin{aligned}
\text{numer} &= \int_0^R \left[\Gamma_1 pr^{-2}\zeta'^2 - 4p(r^{-3}\zeta^2)'\right]\,dr \\
&= \int_0^R \left[\left(\Gamma_1 - \frac{4}{3}\right)pr^{-2}\zeta'^2 + \frac{4}{3}pr^{-2}\zeta'^2 - 8pr^{-3}\zeta\zeta' + 12pr^{-4}\zeta^2\right]\,dr \\
&= \int_0^R \left(\Gamma_1 - \frac{4}{3}\right)pr^{-2}\zeta'^2\,dr + \int_0^R \frac{4}{3}pr^{-2}(\zeta' - 3r^{-1}\zeta)^2\,dr \ge 0,
\end{aligned} \tag{13}$$

with equality only for $\zeta' - 3r^{-1}\zeta = 0$ (equivalently $\zeta \propto r^3$) **and** $\Gamma_1 = \frac{4}{3}$.

Thus we conclude that a Newtonian star with $\Gamma_1 = \frac{4}{3}$ throughout is *linearly neutrally stable*: it possesses a single normal mode with zero frequency, and all other modes have $\sigma^2 > 0$. A full nonlinear analysis is required to understand the true stability (or not) of the star, since one needs to know whether nonlinear effects stabilize or destabilize the neutral mode. Later we will see that in full GR, a star with $\Gamma_1 = \frac{4}{3}$ is de-stabilized.

A Newtonian star with $\Gamma_1 > \frac{4}{3}$ is absolutely stable against small radial perturbations. Hence we expect white dwarfs (for which $\Gamma_1 = \frac{5}{3}$ in the nonrelativistic-electron limit and drops to $\frac{4}{3}$ in the relativistic limit) to be stable. The onset of inverse-β reactions at high density, which reduce Γ_1 below the ideal Fermi gas value of $\frac{4}{3}$, would be expected to de-stabilize the star.

B. Stars with $\Gamma_1 \approx \frac{4}{3}$

What about stars for which Γ_1 is very nearly $\frac{4}{3}$, but is slightly greater in some layers and less in others? This is not an idle question, since we have seen that it is precisely what occurs in a massive WD. The stability of such stars can be addressed using *time-independent perturbation theory*: one starts with an "unperturbed" Hamiltonian (and inner product) and considers the first-order correction to σ^2 that results when one changes it. To first order, the change in the energy functional when one perturbs the functions P, Q, and W is

$$\delta(\sigma^2) = \delta\mathcal{E}[\zeta] = \frac{\int_0^R (\delta P\,\zeta'^2 - \delta Q\,\zeta^2)\,dr}{\int_0^R W\zeta^2\,dr} - \sigma^2 \frac{\int_0^R \delta W\,\zeta^2\,dr}{\int_0^R W\zeta^2\,dr}. \tag{14}$$

[Note that for an eigenmode one does not need to include $\delta\zeta$ corrections because the energy functional is already stationary.] In the particular case where one perturbs from a $\Gamma_1 = \frac{4}{3}$ star with $\zeta = r^3$, this leads to the result for the nearly neutral mode

$$\sigma^2 = \frac{\int_0^R (9r^4\delta P - r^6\delta Q)\,dr}{\int_0^R \rho r^4\,dr}. \tag{15}$$

This is the perturbation theory result for a nearly Newtonian star with Γ_1 near $\frac{4}{3}$. It is valid for any perturbation δP and δQ, whether arising from equation of state corrections or relativistic corrections.

Here we first consider the Newtonian corrections. Since Γ_1 appears only in P and not Q, we have (see Eq. 8):

$$\delta P = \left(\Gamma_1 - \frac{4}{3}\right)\frac{p}{r^2}. \tag{16}$$

Therefore,

$$\sigma^2 = \frac{\int_0^R 9r^2 p(\Gamma_1 - \frac{4}{3})\,dr}{\int_0^R \rho r^4\,dr}. \tag{17}$$

This indicates that (i) the criterion for stability is whether the pressure-averaged adiabatic index

$$\bar\Gamma_1 = \frac{\int_0^R r^2 p\Gamma_1\,dr}{\int_0^R r^2 p\,dr} \tag{18}$$

exceeds $\frac{4}{3}$; and (ii) that the oscillation frequency is

$$\sigma^2 = 2(3\bar\Gamma_1 - 4)\frac{T}{I}, \tag{19}$$

where I is the moment of inertia

$$I = \frac{8\pi}{3}\int_0^R \rho r^4\,dr \tag{20}$$

and T is the the volume-integral of the pressure

$$T = 4\pi\int_0^R r^2 p\,dr. \tag{21}$$

The virial theorem tells us that

$$T = \frac{1}{3}|\Omega|, \tag{22}$$

where Ω is the gravitational binding energy.

IV. GR TERMS AND POST-NEWTONIAN INSTABILITY

Of course, this is a GR class, so our next move will be to put GR back in. But rather than doing a full analysis of the deeply relativistic regime, we will content ourselves with an understanding of how the leading-order GR correction modifies the stability of stars near $\Gamma_1 = \frac{4}{3}$. We note that for an object of mass M and radius R, we have the orders of magnitude

$$P \sim \frac{M^2}{R^6} \quad \text{and} \quad Q \sim \frac{M^2}{R^8}, \tag{23}$$

and we want to work to the next order, i.e. an additional factor of M/R smaller.

To do this, we need only identify the correct δP and δQ to put in Eq. (15). Comparing to Eq. (8), we see that

$$\delta P_{\rm GR} = \Gamma_1 \frac{p}{r^2} e^{\Lambda+3\Phi} - \Gamma_1 \frac{p}{r^2} \approx \frac{4}{3}\frac{p}{r^2}(\Lambda + 3\Phi) \approx \frac{4}{3}\frac{p}{r^2}\left(\frac{m}{r} + 3\Phi\right) \tag{24}$$

(to lowest order in the potentials, and keeping $\Gamma_1 - \frac{4}{3}$ as also a perturbation) and – using the TOV equations several times –

$$\begin{aligned}
\delta Q_{\rm GR} &= e^{\Lambda+3\Phi}\left[\frac{p'^2}{r^2(\rho+p)} + 4\frac{(\rho+p)(m+2\pi r^3 p)}{r^4(r-2m)}\right] + 4\frac{p'}{r^3} \\
&\approx -4(\Lambda+3\Phi)\frac{p'}{r^3} + e^{\Lambda+3\Phi}\left[\frac{p'^2}{r^2(\rho+p)} + 4\frac{(\rho+p)(m+2\pi r^3 p)}{r^4(r-2m)} + 4\frac{p'}{r^3}\right] \\
&= -4(\Lambda+3\Phi)\frac{p'}{r^3} + e^{\Lambda+3\Phi}\left[\frac{p'^2}{r^2(\rho+p)} - 4\frac{p'}{r^3}\frac{m+2\pi r^3 p}{m+4\pi r^3 p} + 4\frac{p'}{r^3}\right] \\
&= -4(\Lambda+3\Phi)\frac{p'}{r^3} + e^{\Lambda+3\Phi}\left[\frac{p'^2}{r^2(\rho+p)} + 8\pi\frac{p'}{r^3}\frac{r^3 p}{m+4\pi r^3 p}\right] \\
&\approx -4(\Lambda+3\Phi)\frac{p'}{r^3} + \frac{p'^2}{r^2(\rho+p)} + 8\pi\frac{pp'}{m} \\
&\approx \frac{p'}{r^3}\left(-4\Lambda - 12\Phi + \frac{rp'}{\rho+p}\right) - 8\pi\frac{p\rho}{r^2} \\
&\approx -\frac{\rho m}{r^5}\left(-4\frac{m}{r} - 12\Phi - \frac{m}{r}\right) - 8\pi\frac{p\rho}{r^2} \\
&= \frac{\rho m}{r^5}\left(5\frac{m}{r} + 12\Phi\right) - 8\pi\frac{p\rho}{r^2}.
\end{aligned} \tag{25}$$

The numerator of Eq. (15) is then (using many integrations by parts, and Newtonian structure relations since we are already at the leading post-Newtonian order):

$$\begin{aligned}
\text{numer} &= \int_0^R \left[9r^4\frac{4}{3}\frac{p}{r^2}\left(\frac{m}{r} + 3\Phi\right) - \frac{\rho m}{r^5}\left(5\frac{m}{r} + 12\Phi\right)r^6 + 8\pi\frac{p\rho}{r^2}r^6\right] dr \\
&= \int_0^R \left[12pmr + 36p\Phi r^2 - 5\rho m^2 - 12\rho m r\Phi + 8\pi p\rho r^4\right] dr \\
&= \int_0^R \left[12pmr + 12p\Phi(r^3)' - 5\rho m^2 - 12\rho m r\Phi + 2pr^2 m'\right] dr \\
&= \int_0^R \left[12pmr - 12(p'\Phi + p\Phi')r^3 - 5\rho m^2 - 12\rho m r\Phi - 2(p'r^2 + 2pr)m\right] dr \\
&= \int_0^R \left[12pmr - 12(-\rho m r\Phi + pmr) - 5\rho m^2 - 12\rho m r\Phi - 2(-\rho m + 2pr)m\right] dr \\
&= \int_0^R \left[-4pmr - 3\rho m^2\right] dr.
\end{aligned} \tag{26}$$

We thus have a contribution to the normal mode frequency from lowest-order GR corrections:

$$\delta(\sigma^2)_{\rm GR} = -\frac{8\pi}{3I}\int_0^R \left[4pmr + 3\rho m^2\right]\, dr. \tag{27}$$

Overall we find a lowest normal-mode frequency:

$$\sigma^2 = 2(3\bar{\Gamma}_1 - 4)\frac{T}{I} - \frac{8\pi}{3I}\int_0^R \left[4pmr + 3\rho m^2\right]\, dr, \tag{28}$$

implying that for small deviations from $\Gamma_1 = \frac{4}{3}$ the star is stable against radial perturbations if

$$\bar{\Gamma}_1 > \frac{4}{3} + \frac{4\pi}{9T}\int_0^R \left[4pmr + 3\rho m^2\right]\, dr. \tag{29}$$

Using $3T = |\Omega|$, we conclude that the condition is

$$\bar{\Gamma}_1 > \frac{4}{3} + \alpha\frac{M}{R}, \tag{30}$$

where

$$\alpha \equiv \frac{R}{3M|\Omega|}\int_0^R \left[4pmr + 3\rho m^2\right] 4\pi r^2\, dr. \tag{31}$$

Inspection shows that α is of order unity – we have $|\Omega| \sim M^2/R$, $\rho \sim M/R^3$, and $p \sim M^2/R^4$, so the integral is of order M^3. Thus we see that **GR is a destabilizing influence on stars with Γ_1 near the critical value of $\frac{4}{3}$**, and that the critical $\bar{\Gamma}_1$ required for stability exceeds $\frac{4}{3}$ by an amount of order the potential well depth.

It is straightforward to evaluate α for e.g. the uniform-density case, in which:

$$\rho = \frac{3M}{4\pi R^3}, \quad m = \frac{r^3}{R^3}M, \quad \text{and} \quad p = \frac{3M^2}{8\pi R^4}\left(1 - \frac{r^2}{R^2}\right). \tag{32}$$

Using $|\Omega| = \frac{3}{5}M/R$, we then find that (substituting $x = r/R$):

$$\alpha = \frac{5}{9}\int_0^1 \left[\frac{3}{2\pi}x^4(1-x^2) + \frac{9}{4\pi}x^6\right] 4\pi x^2\, dx = \frac{125}{189}. \tag{33}$$

It is easily seen that as M/R becomes > 0.1, as it does for more massive neutron stars, the destabilizing effects of GR become significant. Indeed, this explains the existence of a maximum mass for a star made of ideal neutron gas, despite the fact that such matter has $\Gamma_1 > \frac{4}{3}$.