Lecture XXI: Radial pulsations and stability

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I. OVERVIEW

Having constructed the equilibrium stellar configurations, we are now poised to ask whether they are stable to small perturbations. This leads us to the subject of relativistic stellar pulsations. One may classify such pulsations by their angular dependence: when we investigate the full subject we will find modes of every spherical harmonic index $\ell \geq 0$. However, for the cold stars of interest to us the only unstable objects are those with purely radial ($\ell = 0$) unstable modes. In particular, the sequence of stars in between white dwarfs and neutron stars (whose cores are made of the mixed nuclei-in-neutron-gas phase) is unstable against radial oscillations: if infinitesimally perturbed, such an object either implodes to a neutron star, or explodes into a cloud of neutron-rich radioactive matter.

Reading:

- MTW Ch. 26.

II. THE PERTURBATION PROBLEM

We consider for the moment only *linear perturbation theory* for simplicity – second order terms in the perturbations are to be neglected. We also use the overdot $\dot{}$ to denote ∂/∂_t, and the prime $'$ to denote $\partial/\partial r$.

We consider a spherically symmetric, but time-dependent system with metric:

$$ds^2 = -e^{2\Phi}\,dt^2 + e^{2\Lambda}\,dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2), \tag{1}$$

where now both Φ and Λ are functions of (t, r). They may be described by the background solution plus a perturbation:

$$\Phi(t, r) = \Phi_0(t) + \delta\Phi(t, r), \tag{2}$$

and similarly for Λ. The fluid material under consideration also has perturbations in its density ρ, pressure p, and baryon number density n, e.g.

$$\rho(t, r) = \rho_0(t) + \delta\rho(t, r). \tag{3}$$

Finally there is a 4-velocity described by u^t and u^r. We describe such a velocity by the displacement of a fluid parcel. That is, a parcel that in the unperturbed problem was located at (r_o, θ_o, ϕ_o) moves to $(r_o + \xi, \theta_o, \phi_o)$, where the radial coordinate displacement ξ is a function of r_o and t. Since we are in linear perturbation theory it is not necessary to distinguish $\xi(t, r)$ from

$$\xi(t, r_o) = \xi(t, r - \xi) = \xi(t, r) - \xi'(t, r)\xi(t, r) + ..., \tag{4}$$

although in most higher-order perturbation theory analyses ξ is defined in Lagrangian coordinates, i.e. $\xi(t, r_o)$ is taken as the fundamental variable. The 4-velocity of a fluid parcel is then described by the two conditions:

$$\boldsymbol{u}\cdot\boldsymbol{u} = -1 \quad \rightarrow \quad u^t = e^{-\Phi} + \text{h.o.t.} \tag{5}$$

and

$$\frac{u^r}{u^t} = \dot{\xi} \quad \rightarrow \quad u^r = e^{-\Phi}\dot{\xi} + \text{h.o.t.} \tag{6}$$

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A variable such as δp refers to the perturbation in pressure at a given coordinate r. It is thus called an *Eulerian perturbation*. We may also consider the *Lagrangian perturbation* Δp in which one takes as the reference not the background value at that coordinate but the background value at the equilibrium position of the fluid parcel. This differs by

$$\Delta p(t,r) = p(t,r) - p_o(r_o) = p(t,r) - p_o(r - \xi(t,r)) = p(t,r) - p_o(r) + p_o'(r)\xi(t,r) = \delta p(t,r) + p_o'(r)\xi(t,r). \quad (7)$$

A similar transformation holds for $\Delta\rho$, Δn, etc.

With the help of these relations for $\boldsymbol{u}$, the entire problem can be described in terms of the 6 perturbation variables $\delta\Phi$, $\delta\Lambda$, $\delta\rho$, δn, δp, and ξ, all of which are functions of t and r. Our investigations will show that of this set only 2 initial conditions need to be specified, and that the normal modes of oscillation can be found by solving a 2nd order ODE.

A. Background equations

The background solution is as previously derived (with some slight algebraic manipulation):

$$\begin{aligned} \Lambda_o' &= \frac{1}{2r}(1 - e^{2\Lambda_o}) + 4\pi r\rho_o e^{2\Lambda_o}, \\ p_o' &= -(\rho_0 + p_0)\Phi_o', \quad \text{and} \\ \Phi_o' &= -\frac{1}{2r}(1 - e^{2\Lambda_o}) + 4\pi r p_o e^{2\Lambda_o}. \end{aligned} \quad (8)$$

III. THE PERTURBATION EQUATIONS

We now need 6 equations to close the system involving 6 variables. It is not a priori obvious what form the equations will take – as we proceed we will discover that the system is a wave equation in (t,r) with the usual features (hyperbolic system, information travels at the speed of sound, possesses a self-adjoint form, etc.). Of course, this means we will need 6 equations. The ones we will use are:

- Baryon conservation.
- Adiabaticity.
- Energy-momentum conservation (2 nontrivial components).
- Einstein equations (2 nontrivial components).

A. Baryon conservation

The first law here is baryon conservation,

$$(nu^\alpha)_{;\alpha} = 0. \quad (9)$$

Using the product rule, we can re-write this as

$$u^\alpha n_{,\alpha} + n u^\alpha{}_{;\alpha} = 0. \quad (10)$$

The first term is the proper time derivative of n, $dn/d\tau$, along the trajectory of a fluid parcel, which is equal to $d\Delta n/d\tau$. Thus:

$$\frac{d}{d\tau}\Delta n = -n u^\alpha{}_{;\alpha}. \quad (11)$$

Now we see that

$$u^\alpha{}_{;\alpha} = u^\alpha{}_{,\alpha} + \Gamma^\alpha_{\alpha\beta}u^\beta = \dot{u}^t + u^{r\prime} + \Gamma^\alpha_{\alpha t}u^t + \Gamma^\alpha_{\alpha r}u^r. \quad (12)$$

The Christoffel symbols are:

$$\Gamma^{\alpha}_{\alpha t} = \dot{\Phi} + \dot{\Lambda} \quad \text{and} \quad \Gamma^{\alpha}_{\alpha r} = 2r^{-1} + \Phi' + \Lambda', \tag{13}$$

so we have

$$u^{\alpha}{}_{;\alpha} = \partial_t[e^{-\Phi}] + [e^{-\Phi}\dot{\xi}]' + (\dot{\Phi} + \dot{\Lambda})e^{-\Phi} + (2r^{-1} + \Phi' + \Lambda')e^{-\Phi}\dot{\xi} = e^{-\Phi}(\dot{\xi}' + \dot{\Lambda} + 2r^{-1}\dot{\xi} + \Lambda'\dot{\xi}). \tag{14}$$

Now in Eq. (11) it is clear that since Δn is already a perturbation variable, we may use the unperturbed operator $d/d\tau \to e^{-\Phi}\partial_t$. This gives

$$\dot{\Delta n} = -n_0[\dot{\xi}' + \dot{\Lambda} + 2r^{-1}\dot{\xi} + \Lambda'\dot{\xi}]. \tag{15}$$

Integrating, and using that $\Delta n = 0$ in the unperturbed state, we find

$$\Delta n = -n_0 \left[\xi' + \left(\frac{2}{r} + \Lambda'\right)\xi + \delta\Lambda\right] \tag{16}$$

or

$$\Delta n = -n_0[r^{-2}e^{-\Lambda_o}(r^2 e^{\Lambda_o}\xi)' + \delta\Lambda]. \tag{17}$$

Note that this is an initial-value equation – it contains no dynamical information (no dots)!

B. Adiabaticity

The equation of state gives us a relation between the baryon density perturbation and the pressure perturbation assuming no heating of the material (in our case, that it remains cold). This is determined by the adiabatic exponent:

$$\Gamma_1 \equiv \left.\frac{\partial \ln p}{\partial \ln n}\right|_s, \tag{18}$$

where s is the entropy per baryon (zero for cold matter). One then has

$$\Delta p = \Gamma_1 \frac{p_o}{n_o}\Delta n. \tag{19}$$

Substituting in Eq. (17), and converting to an Eulerian perturbation, we find

$$\delta p = -\Gamma_1 p_o[r^{-2}e^{-\Lambda_o}(r^2 e^{\Lambda_o}\xi)' + \delta\Lambda] - p_o'\xi. \tag{20}$$

C. Energy conservation

The conservation of energy in its thermodynamic form tells us that if we follow a fluid parcel along its world line:

$$\frac{d\rho}{dn} = \frac{\rho + p}{n}. \tag{21}$$

The proof of the thermodynamic relations from conservation of the stress-energy tensor is on the homework and I assume you've done that problem already. So this means that

$$\Delta\rho = \frac{\rho_o + p_o}{n_o}\Delta n. \tag{22}$$

Thus, in analogy to our pressure equation:

$$\delta\rho = -(\rho_o + p_o)[r^{-2}e^{-\Lambda_o}(r^2 e^{\Lambda_o}\xi)' + \delta\Lambda] - \rho_o'\xi. \tag{23}$$

D. Einstein equations

We are now interested in the Einstein equations. We write these in terms of the orthonormal basis vectors,

$$\boldsymbol{e}_{\hat{t}} = e^{-\Phi}\boldsymbol{e}_t, \quad \boldsymbol{e}_{\hat{r}} = e^{-\Lambda}\boldsymbol{e}_r, \quad \boldsymbol{e}_{\hat{\theta}} = \frac{1}{r}\boldsymbol{e}_\theta, \quad \text{and} \quad \boldsymbol{e}_{\hat{\phi}} = \frac{1}{r\sin\theta}\boldsymbol{e}_\phi. \tag{24}$$

There are 4 nontrivial components – the $\hat{t}\hat{t}$, $\hat{t}\hat{r}$, $\hat{r}\hat{r}$, and $\hat{\theta}\hat{\theta}$ components (the $\hat{\phi}\hat{\phi}$ part is equivalent to $\hat{\theta}\hat{\theta}$ by symmetry). However with the help of stress-energy conservation, we only need two. We choose $G_{\hat{r}\hat{r}}$ and $G_{\hat{t}\hat{t}}$. [Note: MTW uses $G_{\hat{r}\hat{r}}$ and $G_{\hat{r}\hat{t}}$, but the choice here requires fewer computations.]

Now at this stage you are probably suspecting that for general time-dependent Φ and Λ the Einstein tensor must be horribly complicated. Fortunately, this is not the case. The reason depends on the fact that the Einstein tensor components, because of the way they are constructed from Christoffel symbols, have no more than two derivatives of the metric in any term (one can have terms linear in $g_{\mu\nu,\alpha\beta}$ and terms quadratic in $g_{\mu\nu,\alpha}$).

1. *The $G_{\hat{r}\hat{r}}$ equation*

Let's consider $G_{\hat{r}\hat{r}}$ first. We already know the answer for time-independent Φ and Λ:

$$G_{\hat{r}\hat{r}}|_{\rm static} = 2r^{-1}e^{-2\Lambda}\Phi' + r^{-2}(e^{-2\Lambda} - 1). \tag{25}$$

Allowing these to be time-dependent, one has the possibility of new terms (i) linear in $\ddot{\Phi}$ or $\ddot{\Lambda}$; (ii) quadratic in $\dot{\Phi}$ and $\dot{\Lambda}$; or (iii) linear in $\dot{\Phi}$ or $\dot{\Lambda}$ and possibly a spatial derivative. Of these, (ii) can be neglected in linear perturbation theory, and (iii) cannot contribute to $G_{\hat{r}\hat{r}}$ because they flip sign under $t \rightarrow -t$ whereas $G_{\hat{r}\hat{r}}$ is unaffected.

It turns out there can also be no terms in $G_{\hat{r}\hat{r}}$ linear in $\ddot{\Phi}$, since if we take a perturbation of the form $\delta\Phi(t,r) = \delta\Phi(t)$ (i.e. r-independent) this is equivalent to a redefinition of the time coordinate, which leaves the spacetime and the vector $\boldsymbol{e}_{\hat{r}}$ unchanged and hence implies no change in $G_{\hat{r}\hat{r}}$. One is thus left only with the possibility of a $\ddot{\Lambda}$ term as the only legal modification to Eq. (25):

$$G_{\hat{r}\hat{r}} = 2r^{-1}e^{-2\Lambda}\Phi' + r^{-2}(e^{-2\Lambda} - 1) + f_1(r,\Lambda,\Phi)\ddot{\Lambda}. \tag{26}$$

We thus need only determine the coefficient $f_1(r,\Lambda,\Phi)$ of the $\ddot{\Lambda}$ term. To do this, we recall the general rule:

$$\begin{aligned}
G_{\hat{r}\hat{r}} &= R_{\hat{r}\hat{r}} - \frac{1}{2}(-R_{\hat{t}\hat{t}} + R_{\hat{r}\hat{r}} + R_{\hat{\theta}\hat{\theta}} + R_{\hat{\phi}\hat{\phi}}) \\
&= \frac{1}{2}(R_{\hat{t}\hat{t}} + R_{\hat{r}\hat{r}} - R_{\hat{\theta}\hat{\theta}} - R_{\hat{\phi}\hat{\phi}}) \\
&= \frac{1}{2}(R^{\hat{r}}{}_{\hat{t}\hat{r}\hat{t}} + R^{\hat{\theta}}{}_{\hat{t}\hat{\theta}\hat{t}} + R^{\hat{\phi}}{}_{\hat{t}\hat{\phi}\hat{t}} + R^{\hat{t}}{}_{\hat{r}\hat{t}\hat{r}} + R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{r}} + R^{\hat{\phi}}{}_{\hat{r}\hat{\phi}\hat{r}} - R^{\hat{t}}{}_{\hat{\theta}\hat{t}\hat{\theta}} - R^{\hat{r}}{}_{\hat{\theta}\hat{r}\hat{\theta}} - R^{\hat{\phi}}{}_{\hat{\theta}\hat{\phi}\hat{\theta}} - R^{\hat{t}}{}_{\hat{\phi}\hat{t}\hat{\phi}} - R^{\hat{r}}{}_{\hat{\phi}\hat{r}\hat{\phi}} - R^{\hat{\theta}}{}_{\hat{\phi}\hat{\theta}\hat{\phi}}) \\
&= R^{\hat{\theta}}{}_{\hat{t}\hat{\theta}\hat{t}} + R^{\hat{\phi}}{}_{\hat{t}\hat{\phi}\hat{t}} - R^{\hat{\theta}}{}_{\hat{\phi}\hat{\theta}\hat{\phi}}.
\end{aligned} \tag{27}$$

Now when constructing the Riemann tensor for a diagonal metric, terms involving $\ddot{\Lambda}$ can only be associated with a particular second derivative term in the metric, $g_{rr,tt}$, which can only show up in Riemann tensor components with indices that are some permutation of $rrtt$. No such terms appear in the above, so the coefficient of $\ddot{\Lambda}$ is zero, and we find:

$$G_{\hat{r}\hat{r}} = 2r^{-1}e^{-2\Lambda}\Phi' + r^{-2}(e^{-2\Lambda} - 1). \tag{28}$$

The $\hat{r}\hat{r}$ component of the stress-energy tensor is

$$T_{\hat{r}\hat{r}} = p + (\rho + p)u_{\hat{r}}u_{\hat{r}} = p + \text{h.o.t.} \tag{29}$$

so we find

$$2r^{-1}e^{-2\Lambda}\Phi' + r^{-2}(e^{-2\Lambda} - 1) = 8\pi p. \tag{30}$$

Taking the variation and using Eq. (20) gives

$$2r^{-1}e^{-2\Lambda_o}(\delta\Phi' - 2\Phi_o'\delta\Lambda) - 2r^{-2}e^{-2\Lambda_o}\delta\Lambda = -8\pi\Gamma_1 p_o[r^{-2}e^{-\Lambda_o}(r^2e^{\Lambda_o}\xi)' + \delta\Lambda] - 8\pi p_o'\xi. \tag{31}$$

2. The $G_{\hat{t}\hat{t}}$ equation

The same symmetry-based logic that we applied to $G_{\hat{r}\hat{r}}$ also applies to $G_{\hat{t}\hat{t}}$. We immediately find:

$$G_{\hat{t}\hat{t}} = 2r^{-1}e^{-2\Lambda}\Lambda' - r^{-2}(e^{-2\Lambda} - 1) + f_2(r, \Lambda, \Phi)\ddot{\Lambda}. \tag{32}$$

We can find the $\ddot{\Lambda}$ term using the equation analogous to Eq. (27):

$$G_{\hat{t}\hat{t}} = R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{r}} + R^{\hat{\phi}}{}_{\hat{r}\hat{\phi}\hat{r}} + R^{\hat{\theta}}{}_{\hat{\phi}\hat{\theta}\hat{\phi}}. \tag{33}$$

Since there is no appearance of a permutation of $rrtt$, $\ddot{\Lambda}$ cannot enter into $G_{\hat{t}\hat{t}}$ and so $f_2 = 0$. Therefore this Einstein equation is

$$2r^{-1}e^{-2\Lambda}\Lambda' - r^{-2}(e^{-2\Lambda} - 1) = 8\pi\rho. \tag{34}$$

Perturbing, and using Eq. (23), we find

$$2r^{-1}e^{-2\Lambda_o}(\delta\Lambda' - 2\Lambda_o'\delta\Lambda) + 2r^{-2}e^{-2\Lambda_o}\delta\Lambda = -8\pi(\rho_o + p_o)[r^{-2}e^{-\Lambda_o}(r^2e^{\Lambda_o}\xi)' + \delta\Lambda] - 8\pi\rho_o'\xi. \tag{35}$$

It is convenient to move all of the $\delta\Lambda$ terms to one side and the ξ terms to the other. Then multiplication by $\frac{1}{2}re^{2\Lambda_o}$ enables us to isolate the $\delta\Lambda'$ term on the left-hand side:

$$\delta\Lambda' - 2\Lambda_o'\delta\Lambda + r^{-1}\delta\Lambda + 4\pi(\rho_o + p_o)re^{2\Lambda_o}\delta\Lambda = -4\pi(\rho_o + p_o)r^{-1}e^{\Lambda_o}(r^2e^{\Lambda_o}\xi)' - 4\pi r\rho_o'e^{2\Lambda_o}\xi. \tag{36}$$

To simplify Eq. (36), we consider the function

$$f = \Lambda_o + \Phi_o \quad \rightarrow \quad f' = \Lambda_o' + \Phi_o' = 4\pi r(\rho_o + p_o)e^{2\Lambda_o}. \tag{37}$$

Then the left-hand side simplifies to

$$\delta\Lambda' + \left[\frac{d}{dr}\ln(re^{-2\Lambda_o + f})\right]\delta\Lambda = r^{-1}e^{\Lambda_o - \Phi_o}[re^{\Phi_o - \Lambda_o}\delta\Lambda]'. \tag{38}$$

We therefore have

$$\begin{aligned} [re^{\Phi_o-\Lambda_o}\delta\Lambda]' &= -4\pi(\rho_o + p_o)e^{\Phi_o}(r^2e^{\Lambda_o}\xi)' - 4\pi r^2\rho_o'e^{\Phi_o+\Lambda_o}\xi \\ &= 4\pi e^{\Phi_o+\Lambda_o}[-(\rho_o + p_o)r^2\xi' - (\rho_o + p_o)r^2\Lambda_o'\xi - 2(\rho_o + p_o)r\xi - r^2\rho_o'\xi]. \end{aligned} \tag{39}$$

But as a point of comparison, we see that

$$\begin{aligned} (re^{\Phi_o-\Lambda_o}f'\xi)' &= 4\pi e^{\Phi_o+\Lambda_o}[r^2(\rho_o + p_o)\xi' + 2r(\rho_o + p_o)\xi + r^2(\rho_o + p_o)(\Phi_o' + \Lambda_o')\xi + r^2\rho_o'\xi + r^2p_o'\xi] \\ &= 4\pi e^{\Phi_o+\Lambda_o}[r^2(\rho_o + p_o)\xi' + 2r(\rho_o + p_o)\xi + r^2(\rho_o + p_o)(\Phi_o' + \Lambda_o')\xi + r^2\rho_o'\xi - r^2(\rho_o + p_o)\Phi_o'\xi]. \end{aligned} \tag{40}$$

By inspection this is the opposite of Eq. (39) so

$$[re^{\Phi_o-\Lambda_o}\delta\Lambda]' = -[re^{\Phi_o-\Lambda_o}f'\xi]'. \tag{41}$$

At $r = 0$ both of the objects in brackets are zero, so they are always equal to each other and:

$$\delta\Lambda = -f'\xi = -4\pi r(\rho_o + p_o)e^{2\Lambda_o}\xi. \tag{42}$$

This is the most practical form of the tt Einstein equation.

3. Solution for $\delta\Phi'$

It follows from Eq. (42) that

$$\Phi_o'\delta\Lambda = 4\pi rp_o'e^{2\Lambda_o}\xi. \tag{43}$$

Then Eq. (31) yields

$$2r^{-1}e^{-2\Lambda_o}\delta\Phi' - 2r^{-2}e^{-2\Lambda_o}\delta\Lambda = -8\pi\Gamma_1p_o[r^{-2}e^{-\Lambda_o}(r^2e^{\Lambda_o}\xi)' + \delta\Lambda] + 8\pi p_o'\xi, \tag{44}$$

and substituting Eq. (42) we conclude that

$$\delta\Phi' = -4\pi(\rho_o + p_o)e^{2\Lambda_o}\xi - 4\pi\Gamma_1rp_oe^{2\Lambda_o}[r^{-2}e^{-\Lambda_o}(r^2e^{\Lambda_o}\xi)' - (\Lambda_o' + \Phi_o')\xi] + 4\pi re^{2\Lambda_o}p_o'\xi. \tag{45}$$

Collecting terms gives

$$\delta\Phi' = -4\pi\Gamma_1r^{-1}p_oe^{2\Lambda_o+\Phi_o}(r^2e^{-\Phi_o}\xi)' + 4\pi(rp_o' - \rho_o - p_o)e^{2\Lambda_o}\xi. \tag{46}$$

E. Momentum conservation

We finally consider the momentum conservation law, in the form of the radial component of $T_\mu{}^\nu{}_{;\nu} = 0$. Using the perfect fluid form

$$T_\mu{}^\nu = p\delta_\mu^\nu + (\rho + p)u_\mu u^\nu, \tag{47}$$

we find that

$$p_{,\mu} + (\rho + p)_{,\nu} u_\mu u^\nu + (\rho + p)a_\mu + (\rho + p)u_\mu u^\nu{}_{;\nu} = 0, \tag{48}$$

where the fluid 4-acceleration $a_\mu = u^\nu u_{\mu;\nu}$. Considering the r-component of this equation, we find that to first order the second and fourth terms both vanish (u_r is 1st order and both $u^\nu{}_{;\nu}$ and $u^\nu(\rho + p)_{,\nu}$ must be 1st order since they vanish in the unperturbed configuration), so we are left with

$$p' + (\rho + p)a_r = 0, \tag{49}$$

or raising the index on a:

$$p' + (\rho + p)e^{2\Lambda}a^r = 0. \tag{50}$$

It only remains to find a^r. This can be obtained by following a fluid parcel:

$$a^r = \frac{d^2r}{d\tau^2} + \Gamma^r_{\alpha\beta}u^\alpha u^\beta \tag{51}$$

Now we have $d^2r/d\tau^2 = d^2\xi/d\tau^2$, and since ξ is already 1st-order we may use the background relation $d/d\tau = e^{-\Phi_o}\partial/\partial t$ to set

$$\frac{d^2r}{d\tau^2} = e^{-2\Phi_o}\ddot{\xi}. \tag{52}$$

For the second term, the unperturbed value of u^α is $(e^{-\Phi_o}, 0, 0, 0)$ so we may use the unperturbed value $\Gamma^r_{rt} = 0$ for the $\alpha\beta = rt$ term. Moreover, the perturbed u^t is still $e^{-\Phi}$ to first order. Thus:

$$a^r = e^{-2\Phi_o}\ddot{\xi} + \Gamma^r_{tt}e^{-2\Phi}. \tag{53}$$

We finally have

$$\Gamma^r_{tt} = -\frac{1}{2}e^{-2\Lambda}[-e^{2\Phi}]' = e^{2\Phi-2\Lambda}\Phi', \tag{54}$$

so

$$a^r = e^{-2\Phi_o}\ddot{\xi} + e^{-2\Lambda}\Phi'. \tag{55}$$

Substituting into Eq. (50) gives

$$p' + (\rho + p)e^{2\Lambda-2\Phi_o}\ddot{\xi} + (\rho + p)\Phi' = 0. \tag{56}$$

The linear perturbation of this equation gives

$$\delta p' + (\rho_o + p_o)e^{2\Lambda_o-2\Phi_o}\ddot{\xi} + (\rho_o + p_o)\delta\Phi' + (\delta\rho + \delta p)\Phi'_o = 0. \tag{57}$$

Equation (57) is the evolution equation for ξ. It is second-order, and depends only on the metric and fluid perturbations, which we have seen are all expressible in terms of ξ. Therefore it is a 2nd order (in time) PDE for ξ.

F. Completion of the equation of motion

It remains for us to understand the structure of the equation of motion for the star. This means we must simplify Eq. (57). We begin by noting that

$$r^{-2}e^{-\Lambda_o}(r^2e^{\Lambda_o}\xi)' + \delta\Lambda = r^{-2}e^{-\Lambda_o}(r^2e^{\Lambda_o}\xi)' - (\Lambda'_o + \Phi'_o)\xi = r^{-2}e^{\Phi_o}(r^2e^{-\Phi_o}\xi)', \tag{58}$$

so that

$$\delta\rho = -(\rho_o + p_o)r^{-2}e^{\Phi_o}(r^2 e^{-\Phi_o}\xi)' - \rho_o'\xi \quad \text{and} \quad \delta p = -\Gamma_1 p_o r^{-2}e^{\Phi_o}(r^2 e^{-\Phi_o}\xi)' - p_o'\xi. \tag{59}$$

We define

$$\zeta \equiv r^2 e^{-\Phi_o}\xi \tag{60}$$

and substitute into Eq. (57):

$$\begin{aligned} 0 \;=\; & [-\Gamma_1 p_o r^{-2}e^{\Phi_o}\zeta' - r^{-2}p_o' e^{\Phi_o}\zeta]' + r^{-2}(\rho_o + p_o)e^{2\Lambda_o - \Phi_o}\ddot{\zeta} \\ & +(\rho_o + p_o)[-4\pi\Gamma_1 r^{-1}p_o e^{2\Lambda_o + \Phi_o}\zeta' + 4\pi(rp_o' - \rho_o - p_o)e^{2\Lambda_o + \Phi_o}r^{-2}\zeta] \\ & +[-(\rho_o + p_o + \Gamma_1 p_o)r^{-2}e^{\Phi_o}\zeta' - (\rho_o' + p_o')r^{-2}e^{\Phi_o}\zeta]\Phi_o'. \end{aligned} \tag{61}$$

A further multiplication by $e^{\Lambda_o + 2\Phi_o}$ reduces this to the form

$$0 = W\ddot{\zeta} - (P\zeta')' - Q\zeta, \tag{62}$$

where

$$\begin{aligned} W &= r^{-2}(\rho_o + p_o)e^{3\Lambda_o - \Phi_o}, \\ P &= \Gamma_1 p_o r^{-2}e^{\Lambda_o + 3\Phi_o}, \qquad \text{and} \\ Q &= e^{\Lambda_o + 3\Phi_o}\left[\frac{(p_o')^2}{\rho_o + p_o}r^{-2} - 4p_o' r^{-3} - 8\pi(\rho_o + p_o)p_o r^{-2}e^{2\Lambda_o}\right]. \end{aligned} \tag{63}$$

Note that W, P, and Q are properties of the background and not perturbations.

Here we have used $p_o' = -(\rho_o + p_o)\Phi_o'$ and $4\pi(\rho_o + p_o)re^{2\Lambda_o} = \Lambda_o' + \Phi_o'$ to simplify the ζ' terms, and similar manipulations to simplify Q.

We can thus see the nature of the problem: radial perturbations are described by a wave equation for ζ, with radial coordinate velocity given by

$$c_r^2 = \frac{P}{W} = \frac{\Gamma_1 p_o}{\rho_o + p_o}e^{2\Phi_o - 2\Lambda_o} = c_s^2 e^{2\Phi_o - 2\Lambda_o}, \tag{64}$$

where $c_s^2 = dp/d\rho|_s$ is the conventional adiabatic sound speed.

G. Boundary conditions

No problem is complete without boundary conditions. A wave equation needs two such conditions, one at the inner limit ($r = 0$) and one at the outer limit ($r = R$).

We require that the distortion at the center be finite:

$$\lim_{r\to 0^+}\frac{\zeta}{r} = \text{finite} \tag{65}$$

(zero is allowed but not common). For the outer boundary condition, we require the pressure to remain zero, i.e. $\Delta p = 0$ or

$$\lim_{r\to R^-}(\Gamma_1 p_0 \zeta') = 0. \tag{66}$$

IV. STABILITY AND EIGENMODES

The wave equation, Eq. (62), with boundary conditions is a self-adjoint problem and hence admits a complete set of eigenmodes. That is, the solution is a superposition of solutions of the form

$$\zeta(t, r) = \zeta(0, r)e^{i\sigma t}, \tag{67}$$

where $-\sigma$ is the oscillation frequency. (The use of σ is standard in stellar oscillation theory.) The corresponding eigenvalue problem is

$$\sigma^2 W\zeta = -(P\zeta')' - Q\zeta. \tag{68}$$

For stable matter ($\Gamma_1 > 0$) the "kinetic" term P is positive everywhere. It is also true that Q is positive everywhere; this is more obvious if we substitute the TOV equation in Eq. (63), yielding

$$Q = e^{\Lambda+3\Phi}\left[\frac{p'^2}{r^2(\rho+p)} + 4\frac{(\rho+p)(m+2\pi r^3 p)}{r^4(r-2m)}\right]. \tag{69}$$

Equation (68) for the eigenfrequencies of the star is most easily thought of as like the problem of determining quantum energy levels in a 1D potential using the Schrödinger equation: the difference is that if σ^2 is the energy then the kinetic-like term is positive definite but the potential energy term is negative definite. The problem therefore has a "ground state" (minimum σ^2 mode; breathing mode, ζ has the same sign everywhere) and an infinite hierarchy of modes with successively more radial nodes.

If the perturbation problem admits a negative σ^2 mode then the star is unstable: a slight perturbation causes it to grow or shrink at an ever-expanding rate. One does not expect to find such a star in nature.

A. Variational principle

A common method to understand the stability of a star analytically is to use the *variational principle*. This says that the eigensolutions for Eq. (68) are systems for which the functional

$$\mathcal{E}[\zeta] \equiv \frac{\int_0^R (P\zeta'^2 - Q\zeta^2)\,dr}{\int_0^R W\zeta^2\,dr} \tag{70}$$

is stationary with respect to small perturbations $\delta\zeta$ (prove this!), and that the value of $\mathcal{E}[\zeta]$ is σ^2. The minimum value of the energy functional over all ζ is the "ground state" (least stable or lowest-frequency mode). We therefore have that

$$\text{stability} \quad \leftrightarrow \quad \mathcal{E}[\zeta] > 0 \;\forall\; \zeta(r). \tag{71}$$

[Such principles exist for most stability arguments, even in Newtonian physics.]