Lecture XX: The equation of state of dense matter

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I. OVERVIEW

Thus far we have investigated the relativistic equations of stellar structure and the exterior solutions. Our attention now focuses on the interior: what is the composition of the dense matter in stellar remnants?

This is not really a GR subject but it is so central to the astrophysical applications of GR that I will cover it anyway.

This lecture is based on the following references, which you may find helpful:

- Shapiro & Teukolsky, *Black Holes, White Dwarfs, and Neutron Stars* (1983)
- A more up-to-date reference for lower density is Chamel & Haensel
`http://relativity.livingreviews.org/Articles/lrr-2008-10/`
- And if you want to read about high density equations of state, there is the book by Glendenning, *Compact Stars* (2000)

II. THE KEY EQUATIONS SUMMARIZED

The metric for a spherical star is

$$ds^2 = -e^{2\Phi}\,dt^2 + \frac{dr^2}{1-2m/r} + r^2(d\theta^2 + \sin^2\theta\,d\phi^2), \tag{1}$$

with

$$\frac{dm}{dr} = 4\pi r^2\rho \quad \text{and} \quad \frac{dp}{dr} = -\frac{(m+4\pi r^3 p)(\rho+p)}{r(r-2m)}. \tag{2}$$

We are interested in a star made of cold matter with a particular equation of state $p(\rho)$. The gravitational potential at the surface is

$$\Phi(R) = \frac{1}{2}\ln\left(1 - 2\frac{M}{R}\right), \tag{3}$$

and using the equation of hydrostatic equilibrium, $dp/dr = -(\rho+p)\,d\Phi/dr$, in the interior it is

$$\Phi(r) = \Phi(R) - \int_0^p \frac{dp}{\rho+p}. \tag{4}$$

Normal matter also has a conserved baryon density $n(\rho)$, and correspondingly a conserved baryon current $I^\mu = nu^\mu$. A star then has a baryon density (here dV_μ is the volume vector, which is the normal to a 3-surface Σ times the volume element)

$$N = \int_\Sigma I^\mu\,dV_\mu = \int_0^R n\,\frac{4\pi r^2}{\sqrt{1-2m/r}}\,dr. \tag{5}$$

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A. Thermodynamics

Since baryon number is conserved as material is compressed, at zero temperature the first law of thermodynamics gives us a relation of the pressure, (energy) density, and baryon density. Remembering that the volume per baryon is n^{-1}, and the energy per baryon is ρ/n, we have

$$d(\rho n^{-1}) = -p\, d(n^{-1}). \tag{6}$$

Expanding this and solving for dn, we find

$$\frac{dn}{n} = \frac{d\rho}{p+\rho} \tag{7}$$

and hence

$$n = n_0 \exp \int_{\rho_0}^{\rho} \frac{d\rho}{p+\rho}, \tag{8}$$

where n_0 and ρ_0 are the baryon and energy densities at zero pressure (corresponding to a perfect crystal of ^{56}Fe). Alternatively if the relation between ρ and n can be calculated then we easily see that

$$p = n^2 \frac{d}{dn}\left(\frac{\rho}{n}\right). \tag{9}$$

Often the functions $\rho(n)$ and hence $p(n)$ are most easily computed.

III. THE PROPERTIES OF COLD MATTER AT LOW DENSITY

We first consider the case of low density, i.e. $\rho \lesssim 10^6$ g/cm^3. At such densities the ground state of baryonic matter is composed of the ^{56}Fe nuclei, plus electrons arranged in various configurations. In fact real stars in this density range (white dwarfs) do not achieve full nuclear equilibrium – the hydrogen from which their progenitors formed has burned at least to ^{4}He, and more often to the products of helium burning (^{12}C/^{16}O) or even further (^{16}O/^{20}Ne). But stars massive enough to burn their cores to iron have a more violent fate in store than white dwarf formation. We will therefore keep the composition (atomic number Z and mass number A) free here, and remember that the idealized ground state of matter is $(Z, A) = (26, 56)$.

A. Degenerate electron gases

The simplest case to study is the equation of state of matter in which electrostatic forces play no role. We'll do the calculations for this case and then consider its range of validity.

The density of matter can be written as the sum of the electrons and the nuclei since they are taken as noninteracting. At a given baryon density n, the number density of nuclei is n/A, and hence the corresponding energy density in the nuclei is

$$\rho_{\rm n} = \frac{n}{A}[Zm_p + (A-Z)m_n - B], \tag{10}$$

where B is the binding energy of the nucleus.

For the lighter electrons, the degeneracy pressure is important. We may obtain the electron density as $n_e = (Z/A)n$, and then obtain the Fermi momentum by phase space considerations. Setting $\hbar = 1$ (Planck units!) we find

$$n_e = \frac{2}{(2\pi)^3}\frac{4}{3}\pi p_{\rm F}^3, \tag{11}$$

where the factor of 2 comes from electron spin and $p_{\rm F}$ is the *Fermi momentum*. We define the electron density factor as

$$x \equiv \frac{p_{\rm F}}{m_e} = \frac{(3\pi^2 n_e)^{1/3}}{m_e} = \left(3\pi^2\frac{Z}{A}\right)^{1/3}\frac{n^{1/3}}{m_e}, \tag{12}$$

which is related to n via

$$n = \frac{A}{Z} m_e^3 \frac{x^3}{3\pi^2}. \tag{13}$$

The energy density in the electrons is

$$\rho_e = \int_0^{p_{\rm F}} \sqrt{m_e^2 + p^2}\, \frac{2}{(2\pi)^3} 4\pi p^2\, dp = \frac{m_e^4}{\pi^2} \int_0^x \sqrt{1+y^2}\, y^2\, dy, \tag{14}$$

where we have substituted $p = m_e y$. The integral is tedious (substitute $y = \sinh\alpha$ and use either double-angle identities or reduce to exponentials) but evaluates to

$$\rho_e = m_e^4 \chi(x), \tag{15}$$

where

$$\chi(x) = \frac{1}{8\pi^2} \left\{ x(1+2x^2)\sqrt{1+x^2} - \ln\left[x + \sqrt{1+x^2}\right] \right\} = \frac{x^3}{3\pi^2} + \frac{x^5}{10\pi^2} + ... \tag{16}$$

We thus find a total density of

$$\rho = \frac{n}{A}[Z m_p + (A-Z)m_n - B] + m_e^4 \chi(x). \tag{17}$$

An alternate way to write this, useful at low densities, is to extract the electron rest mass from the χ integral, i.e. to write

$$\rho = \frac{n}{A}[Z(m_p + m_e) + (A-Z)m_n - B] + m_e^4 \left[\chi(x) - \frac{x^3}{3\pi^2}\right]. \tag{18}$$

We note that the last term here is usually small if x is not too large ($x \ll m_p/m_e$), in which case the energy density is nearly the rest mass energy of the particles involved, i.e. $\rho \approx n m_{\rm B}$ where $m_{\rm B} = m(^{12}C)/12$ is the atomic mass unit.

The pressure in this case is easily derived from Eq. (9). We see that

$$\frac{\rho}{n} = \frac{1}{A}[Z(m_p + m_e) + (A-Z)m_n - B] + m_e \left[\frac{3\pi^2}{x^3}\chi(x) - 1\right]. \tag{19}$$

Using $n\, d/dn = \frac{1}{3} x\, d/dx$, we find

$$p = \frac{1}{3} n m_e x \frac{d}{dx}\left(\frac{3\pi^2}{x^3}\chi(x)\right) = \frac{1}{3} m_e^4 x^4 \frac{d}{dx}\frac{\chi(x)}{x^3}. \tag{20}$$

The differentiation is tedious but leads to

$$p = m_e^4 \phi(x), \tag{21}$$

where

$$\phi(x) = \frac{x^4}{3}\frac{d}{dx}\frac{\chi(x)}{x^3} = \frac{1}{8\pi^2}\left\{ x\left(\frac{2}{3}x^2 - 1\right)\sqrt{1+x^2} + \ln\left[x + \sqrt{1+x^2}\right] \right\}. \tag{22}$$

At low densities ($x \ll 1$) we have $\phi(x) \approx x^5/(15\pi^2)$.

The degenerate electron gas has two limits for its equation of state:

- $x \ll 1$: Here $\phi(x) \to x^5/(15\pi^2)$ and $\chi(x) \to x^3/(2\pi^2)$. In this case we have $p \propto x^5 \propto \rho^{5/3}$.
- $x \gg 1$: Here $\phi(x) \to x^4/(12\pi^2)$ and $\chi(x) \to x^4/(4\pi^2)$. In this case we have $p \propto x^4 \propto \rho^{4/3}$.

In conventional (cgs) units, we have

$$p = 1.42 \times 10^{25} \phi(x)\,\text{dyne/cm}^2, \quad n_e = 5.95 \times 10^{29} x^3\,\text{cm}^{-3}, \quad \text{and} \quad \rho = 9.7 \times 10^5 \frac{A}{Z} x^3\,\text{g/cm}^3. \tag{23}$$

In the core of a white dwarf, we typically have $x \sim \mathcal{O}(1)$. The pressure is then $\sim 10^{25}$ dyne/cm^2, or 10^{19} atmospheres!

B. The small-x limit – electrostatic corrections

Thus far, we have neglected the electrostatic energy of the particles. If we imagine each nucleus occupying a cubical region of size $a \sim (A/n)^{1/3}$ and having charge Z, then by organizing the nuclei in a self-avoiding fashion one finds an electrostatic correction to the energy density of order the Coulomb energy:

$$\Delta\rho_{\rm es}(n) \sim -\frac{(Ze)^2}{a^4} \sim -\frac{\alpha Z^2}{A^{4/3}} n^{4/3} \sim -0.0158\alpha Z^{2/3} m_e^4 x^4, \tag{24}$$

where the prefactor is for the lowest energy configuration – a body centered cubic lattice (see S&T). [Whether a lattice forms depends on the temperature – at truly $T = 0$, as we are considering, a solid is forms, but it melts if the temperature exceeds a melting temperature given by the electrostatic energy per nucleus: $\sim 0.01\alpha Z^2 (n/A)^{1/3}$. The lattice energy, while important for cooling and perhaps white dwarf seismology, is irrelevant for the overall structure or equation of state.]

The electrostatic correction is negative since the rearrangement leads to a reduction in energy. This is a small correction to the energy density of the material, but a potentially big correction to the pressure:

$$\Delta p_{\rm es}(n) = n \frac{d}{d\ln n} \frac{\Delta\rho_{\rm es}(n)}{n}; \tag{25}$$

since $\Delta\rho_{\rm es}(n)/n \propto n^{4/3}/n \propto n^{1/3}$, we have

$$\Delta p_{\rm es}(n) = \frac{1}{3}\Delta\rho_{\rm es}(n) \sim -0.0053\alpha Z^{2/3} m_e^4 x^4. \tag{26}$$

Our neglect of electrostatic interactions is thus valid only in the limit of $|\Delta p_{\rm es}| \ll p$, or

$$0.0053\alpha Z^{2/3} m_e^4 x^4 \ll \frac{m_e^4 x^5}{15\pi^2}, \tag{27}$$

or

$$x \gg x_c \sim 0.78\alpha Z^{2/3}. \tag{28}$$

The characteristic density at which electrostatic interactions dominate is then (see Eq. 23):

$$\rho_c = 9.7 \times 10^5 \frac{A}{Z} (0.78\alpha Z^{2/3})^3 \ {\rm g/cm^3} = 0.2 AZ \ {\rm g/cm^3} \tag{29}$$

and the implied characteristic pressure is

$$p_c = 1.42 \times 10^{25} \frac{(0.78\alpha Z^{2/3})^5}{15\pi^2} \ {\rm dyne/cm^2} = 6 \times 10^{11} Z^{10/3} \ {\rm dyne/cm^2}. \tag{30}$$

A cold sample of material at pressures below p_c transforms from the degenerate state into discrete atoms (or ions), since the electrostatic binding energy of electrons to the nuclei exceeds the Fermi energy.

The pressure in a white dwarf (and certainly in a neutron star) greatly exceeds p_c and so we will not consider the electrostatic effects further. [Aside: The internal pressure in a large planet such as Jupiter reaches $\sim 10^{14}$ dyne/cm^2 $> p_c$, and so deep in its interior the hydrogen atoms are actually crushed and turn into a liquid metallic state.]

C. High densities

As one increases the density of matter, the ground state may no longer be a ^{56}Fe crystal. In nuclear equilibrium, we expect the composition to change to minimize ρ subject to fixed n, but Z and A may vary. This may involve fusion or fission of the nuclei (or other rearrangements), as well as β-reactions such as

$$(Z, A) + e^- \to (Z-1, A) + \nu_e \tag{31}$$

and

$$(Z, A) \to (Z+1, A) + e^- + \bar{\nu}_e. \tag{32}$$

How is one to find the "best" Z and A? At a given pressure p (and hence a given x), one may minimize the Gibbs free energy per baryon, which at zero temperature is

$$G = \frac{\rho + p}{n} = \frac{Z}{A}m_p + \frac{A-Z}{A}m_n - \frac{B}{A} + m_e^4 \frac{\chi(x) + \phi(x)}{n}; \tag{33}$$

considering the form of χ and ϕ, and using the substitution that $n = m_e^3(A/Z)x^3/(3\pi^2)$, we find

$$G = \frac{Z}{A}m_p + \frac{A-Z}{A}m_n - \frac{B}{A} + \frac{Z}{A}m_e\sqrt{1+x^2}. \tag{34}$$

The meaning of this equation is simple: the preferred nucleus is the one with the lowest mass per baryon, including binding energy as well as the Fermi energy (not just rest mass energy!) of the electron required for charge conservation. This is not a surprise.

[Really we should include the Coulomb corrections in this calculation, but they don't affect the qualitative results.]

If x is small, the "penalty" for adding an electron is simply m_e, and the ground state is ^{56}Fe. However as one increases the density and the penalty for an electron increases, neutron-rich nuclei are favored: while they are much more massive than symmetric nuclei of the same mass number, the electron Fermi energy prevents them from β-decaying.

The first such transition is to ^{62}Ni ($Z = 28$) at $x = 1.6$ or $\rho = 8 \times 10^6$ g/cm^3. From there comes a sequence of successively more neutron-rich nuclei:

$$^{56}\text{Fe} \rightarrow {}^{62}\text{Ni} \rightarrow {}^{64}\text{Ni} \rightarrow {}^{66}\text{Ni} \rightarrow {}^{86}\text{Kr} \rightarrow ... \rightarrow {}^{118}\text{Kr}, \tag{35}$$

where ^{118}Kr forms at $\rho = 3.8 \times 10^{11}$ g/cm^3.

Computation of this sequence requires a detailed model for the binding energy B of a nucleus. However, the qualitative features can be described by the liquid-drop model, in which the binding energy is modeled as:

$$B = b_1 A - b_2 A^{2/3} - b_4 A\left(\frac{1}{2} - \frac{Z}{A}\right)^2 - b_5\frac{Z^2}{A^{1/3}}, \tag{36}$$

where the b_i are positive coefficients and the terms refer to:

- The binding energy from the short-range attractive nuclear force, which provides a contribution proportional to the nuclear volume or number of nucleons.
- The surface tension, proportional to surface area, since nucleons at the surface have fewer neighbors.
- The symmetry energy, a repulsive contribution if the density of protons and neutrons is different.
- The electrostatic repulsion, proportional to charge squared over radius.

[Better agreement with experimental binding energies occurs if one adds odd-even terms and closed-shell terms to account for the discrete nature of protons and neutrons. But this does not affect our qualitative picture.]

What is of note here is that as the nucleus becomes more neutron-rich, the optimal mass increases, since the Coulomb term (the only term that disfavors conglomeration of nuclei into bigger and bigger objects; indeed the one that is responsible for fission) becomes less important. This is responsible for the formation of higher-mass nuclei in Eq. (35).

But what is most important about Eq. (36) is that when the nucleus becomes sufficiently asymmetric, even neutrons are unbound. To be explicit:

$$\frac{\partial B}{\partial A} = b_1 - \frac{2}{3}b_2 A^{-1/3} - 2b_4\left(\frac{1}{4} - \frac{Z^2}{A^2}\right) + \frac{1}{3}b_5\frac{Z^2}{A^{4/3}}. \tag{37}$$

When the nucleus becomes sufficiently asymmetric, the b_4 term can drive the marginal binding energy of a neutron negative: $\partial B/\partial A < 0$. [Remark: $b_4/b_5 \sim 130$.] In this case, the neutrons become unbound and a sea of free neutrons forms in between the nuclei, leading to *neutron drip*. This occurs at $\rho \sim 4 \times 10^{11}$ g/cm^3 ($p \sim 10^{30}$ dyne/cm^2), i.e. $x = 50$ and the electron Fermi energy is 25 MeV.

IV. DENSITIES EXCEEDING NEUTRON DRIP

At densities exceeding neutron drip, a neutron gas forms outside the nuclei. Above a few times the drip density, most of the nuclear matter is actually in free nucleons, and so to a first approximation one constructs an equation of state from a Fermi-degenerate gas of electrons, protons, and neutrons. Be warned however that the protons are in fact bound into individual nuclei, with the (larger) number of neutrons forming a gas. Due to the large Fermi energy of the electrons we neglect the neutron-proton mass difference.

If this gas is degenerate but nonrelativistic ($x_n \ll 1$) and hence there exists a free neutron density

$$n_n = m_n^3 \frac{x_n^3}{3\pi^2}, \tag{38}$$

and a neutron pressure given by

$$p_n = m_n^4 \phi(x_n) \approx m_n^4 \frac{x_n^5}{15\pi^2}, \tag{39}$$

and similar for the protons (with x_p). This is in addition to the pressure associated with the electrons, which in the relativistic electron limit is

$$p_e = m_e^4 \phi(x_e) \approx m_e^4 \frac{x_e^4}{12\pi^2}. \tag{40}$$

Now the β-equilibrium condition is that the Fermi energies of the proton and electron add to that of the neutron:

$$\frac{1}{2} m_n x_p^2 + m_e x_e = \frac{1}{2} m_n x_n^2, \tag{41}$$

so we can see that the neutrons are more abundant. Also charge neutrality gives $n_p = n_e$ or

$$m_n x_p = m_e x_e. \tag{42}$$

We thus see that the neutron density parameter satisfies:

$$\frac{1}{2} x_p^2 + x_p = \frac{1}{2} x_n^2. \tag{43}$$

At $x_n \ll 1$ we thus have $x_p = \frac{1}{2} x_n^2 \ll x_n$, so we conclude that the matter is composed mostly of neutrons. The equation of state is then approximated as an ideal degenerate neutron gas. In this case:

$$\rho \approx 6 \times 10^{15} x_n^3 \,\mathrm{g/cm^3} \quad \text{and} \quad p \approx p_n = 10^{36} x_n^5 \,\mathrm{dyne/cm^2}. \tag{44}$$

The proton pressure is small compared to the neutron pressure since $x_p \ll x_n$, but since $x_e \gg 1$ it is not obvious whether the electron pressure might be significant. It is not:

$$\frac{p_e}{p_n} = \frac{m_e^4 x_e^4/(12\pi^2)}{m_n^4 x_n^5/(15\pi^2)} = \frac{5 m_e^4 (m_p x_p/m_e)^4}{4 m_n^4 x_n^5} = \frac{5 x_p^4}{4 x_n^5} = \frac{5}{64} x_n^3 \ll 1. \tag{45}$$

What is the actual state of nuclear matter in the density range from $\rho_{\rm drip}$ up to nuclear density? While the equation above predicts both protons and electrons to be present, one would expect based on the previous discussion that the protons are clumped into discrete nuclei, with neutron gas in between. This is indeed correct. The basic reason is that nucleons attract and hence tend to cluster into nuclei – but the symmetry energy implies that this is only favored if the proton:neutron ratio in the nuclei is not too extreme. The result is the formation of a multiphase medium with clusters of nuclear density and a dilute neutron-gas background, all pervaded by the electron Fermi gas. The nuclei are positively charged and the background is negative (because of the electrons), so there is a Coulomb energy cost to this arrangement that grows in proportion to the nuclear lattice spacing a as

$$\Delta E_{\rm Coulomb} \propto a^2. \tag{46}$$

[Think about the electric field energy density with blobs of size a and fixed charge density ρ_q – the electric field is proportional to a so its energy cost is $\sim a^2$.] So one might imagine a very small scale lattice structure. However there is a surface tension cost proportional to the surface area:volume ratio,

$$\Delta E_{\rm surface} \propto a^{-1}. \tag{47}$$

This leads to a minimum-energy lattice scale where $2\Delta E_{\rm Coulomb} = \Delta E_{\rm surface}$.

It is interesting to note that Eq. (44) implies a larger $\rho(p)$ at neutron drip than we found based on the treatment with nucleons – the implied density at 10^{30} dynes/cm^2 is 1.5×10^{12} g/cm^3 or $4\rho_{\rm drip}$. The implication is that above neutron drip the adiabatic exponent $\Gamma_1 \equiv d\ln p/d\ln\rho$ must drop from its value of $\frac{4}{3}$ (for the relativistic electron gas) to a lower value (<1) before recovering to the $\frac{5}{3}$ predicted for the neutron gas.

This multiphase structure persists until the density becomes of order nuclear density, $\rho\approx\rho_{\rm nuc}/3 = 10^{14}$ g/cm^3. As nuclear density is approached, the nuclei occupy a significant fraction of the volume. When this occurs, the minimum of the combined Coulomb+surface energy is achieved not with discrete nuclei but with other shapes – rods or slabs of dense nuclear material embedded in the diffuse neutron gas. (Such phases are called *pasta phases.*) At higher densities the topology reverses to slabs, rods, and then droplets of neutron gas in a medium filled with mostly nuclear-density material, until finally at $\rho\sim1.5\times10^{14}$ g/cm^3 ($p\sim5\times10^{32}$ dyne/cm^2) the material becomes a uniform neutron-rich, nuclear-density medium.

Note that at this density the matter is starting to become slightly relativistic: $p/\rho\sim0.004$. Stars with density above this range must be treated using GR for precise calculations.

V. NUCLEAR DENSITIES AND ABOVE

What happens at densities exceeding nuclear? If one takes seriously the degenerate gas of protons, neutrons, and electrons, then at high densities one must treat the protons and neutrons as mildly relativistic. The β-equilibrium result is then

$$m_p\sqrt{1+x_p^2} + m_e x_e = m_n\sqrt{1+x_n^2} \tag{48}$$

or, using $m_e x_e = m_p x_p$:

$$\sqrt{1+x_p^2} + x_p = \sqrt{1+x_n^2}. \tag{49}$$

Since the neutrons dominate, we will use x_n as the basic independent parameter. Then we solve for x_p:

$$x_p = \frac{x_n^2}{2\sqrt{1+x_n^2}}. \tag{50}$$

The baryon density is

$$n = \frac{m_n^3}{3\pi^2}(x_p^3+x_n^3) = \frac{m_n^3}{3\pi^2}x_n^3\left[1+\frac{x_n^3}{8(1+x_n^2)^{3/2}}\right]. \tag{51}$$

The density and pressure of the degenerate gas are

$$\rho = m_n^4\left[\chi(x_p)+\chi(x_n)+\frac{x_p^4}{4\pi^2}\right] \tag{52}$$

and

$$p = m_n^4\left[\phi(x_p)+\phi(x_n)+\frac{x_p^4}{12\pi^2}\right]. \tag{53}$$

The chemical potentials of the species are:

$$\mu_e = m_e x_e = m_n\frac{x_n^2}{2\sqrt{1+x_n^2}}, \quad \mu_p = m_n\frac{2+x_n^2}{2\sqrt{1+x_n^2}}, \quad \text{and} \quad \mu_n = m_n\sqrt{1+x_n^2}. \tag{54}$$

At high density, however, many additional phenomena can occur:

The high-density asymmetric ($n_n > n_p$) fluid has an additional energy cost (ultimately due to vector meson exchange). There is a corresponding modification to the equation of state – it is "stiffer" (more repulsive) than the noninteracting gas.

At $x_n = 0.5$ ($\rho \sim 7 \times 10^{14}$ g/cm^3), this model predicts that the electron chemical potential exceeds the muon mass. Above this density, β-equilibrium predicts that it is favorable for neutrons to decay and begin to fill muon states:

$$n \to p^+ + \mu^- + \bar{\nu}_\mu. \tag{55}$$

The star is then predicted to contain a Fermi gas of muons in addition to electrons, protons, and (mostly) neutrons.

At even higher densities, the neutron chemical potential becomes large enough that we expect the formation of *hyperons* (heavy baryons). For example, the Σ^- baryon (quark content: dds) has mass 1197 MeV and so the reaction

$$n + e^- \to \Sigma^- + \nu_e \tag{56}$$

becomes favorable when

$$m_n(\sqrt{1 + x_n^2} + x_p) > m_\Sigma. \tag{57}$$

The Σ^- particle is normally unstable, decaying by the weak reaction

$$\Sigma^- \to n + e^- + \nu_e, \tag{58}$$

but at high density the exclusion principle blocks the final electron states and stabilizes the Σ^-. Other hyperons such as the Λ^0 (quark content: uds) may also be formed at such densities.

Other particles may also be produced. For example, if the difference of neutron and proton chemical potentials exceeds the pion mass (140 MeV – at least in vacuum!), one might expect the decay:

$$n \to p^+ + \pi^- \tag{59}$$

(quark content of π^-: $d\bar{u}$). The pion has spin 0 and as such is a boson: it is not subject to the exclusion principle. Therefore at zero temperature, rather than forming a degenerate gas, the pions would form a Bose-Einstein condensate. There is doubt, however, about whether this *pion condensation* actually takes place once one considers interactions of the pions with the surrounding medium (which changes their effective mass).

At such high densities, however, one runs into other uncertainties. We have thus far treated hadrons as fundamental particles, but they are of course composed of quarks and gluons. The theory of their interactions is *quantum chromodynamics* (QCD). At the risk of gross oversimplification: our conceptual picture of the QCD vacuum is that it consists of a sea of quark-antiquark pairs that exclude color-electric fields, in much the same way that a superconductor excludes magnetic fields. A neutron consists of 3 quarks trapped in a bubble (or "bag") of restored QCD vacuum (without the pairs) that allows such fields. In the phenomenological *MIT bag model*, the size of a neutron is determined by the balance between the degeneracy pressure of the trapped quarks (which tends to inflate the bag since it is proportional to $r_{\rm bag}^{-1}$) versus the energy cost of the restored vacuum inside (proportional to $r_{\rm bag}^3$). Such a model predicts that at densities where the neutrons overlap (several times $\rho_{\rm nuc}$) there might be a phase transition to *quark matter*, where there is restored vacuum everywhere and the Fermi surface calculations should be done with u, d, and s quarks instead of baryons. Such a transition could even include pasta-like phases (mixed hadronic/quark) with different charges in the various phases.

At truly high densities one might imagine heavier quarks such as the charm – but if so, this would be at much higher density ($\sim 10^{17}$ g/cm^3), since the u quarks would have to have Fermi energy at the charm mass. We don't expect these densities even in a neutron star, so we conclude our investigation here.