Lecture XIX: Particle motion exterior to a spherical star

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(Dated: January 18, 2012)

I. OVERVIEW

Our next objective is to consider the motion of test particles exterior to a spherical star (or around a black hole), i.e. in the Schwarzschild spacetime. We are interested in both massive particles (e.g. Mercury orbiting the Sun, or a neutron star orbiting a supermassive black hole), and in massless particles (light rays carrying information from an astrophysical object).

The reading for this lecture is:

- MTW Ch. 25.

II. THE PROBLEM

We begin with the metric:

$$ds^2 = -\left(1 - 2\frac{M}{r}\right) dt^2 + \frac{dr^2}{1 - 2M/r} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{1}$$

Without loss of generality, it is permissible to assume the particle orbits in the equatorial plane, $\theta = \pi/2$. In this case, we only need the 2+1 dimensional equatorial slice $\mathbb{E} \subset \mathcal{M}$ of the metric:

$$ds^2 = -\left(1 - 2\frac{M}{r}\right) dt^2 + \frac{dr^2}{1 - 2M/r} + r^2\, d\phi^2. \tag{2}$$

The particle's momentum $\boldsymbol{p}$ has 3 nontrivial components, and fortunately has three conserved quantities: the time-translation and rotation (longitude invariance or $\boldsymbol{J}_3$ Killing field) imply that p_t and p_ϕ are conserved. Furthermore, the magnitude of the 4-momentum is conserved, $\boldsymbol{p} \cdot \boldsymbol{p} = -\mu^2$. The existence of 3 conserved quantities implies that the motion of the test particle is integrable.

III. MOTION OF MASSIVE PARTICLES

Let us first consider the motion of a massive particle. We may then define the *specific energy* $\tilde{E}$ and *specific angular momentum* $\tilde{L}$ to be the conserved quantities per unit mass associated with the two symmetries:

$$\tilde{E} = -u_t = -\frac{p_t}{\mu} \quad \text{and} \quad \tilde{L} = u_\phi = \frac{p_\phi}{\mu}. \tag{3}$$

For a particle at rest far from the star ($r \to \infty$), $\tilde{E} \to 1$.

A. The 4-velocity

Using the inverse metric components,

$$g^{tt} = -\frac{1}{1 - 2M/r}, \quad g^{rr} = 1 - 2\frac{M}{r}, \quad \text{and} \quad g^{\phi\phi} = r^{-2}, \tag{4}$$

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we use $g^{\mu\nu}u_\mu u_\nu = -1$ to see that

$$-\frac{(u_t)^2}{1-2M/r} + \left(1-2\frac{M}{r}\right)(u_r)^2 + \frac{(u_\phi)^2}{r^2} = -1. \tag{5}$$

We may then solve for u_r as a function of r:

$$(u_r)^2 = \frac{1}{1-2M/r}\left(-1 + \frac{\tilde{E}^2}{1-2M/r} - \frac{\tilde{L}^2}{r^2}\right); \tag{6}$$

except that we can't determine the sign (we don't know whether the particle is moving inward or outward). It is common also to compute the 4-velocity components with contravariant indices, since these are most closely related to the actual trajectory: $dx^\mu/d\tau = u^\mu$. These are:

$$\begin{aligned} u^t &= \frac{\tilde{E}}{1-2M/r}, \\ u^r &= \pm\sqrt{\left(1-2\frac{M}{r}\right)\left(-1+\frac{\tilde{E}^2}{1-2M/r}-\frac{\tilde{L}^2}{r^2}\right)}, \quad \text{and} \\ u^\phi &= \frac{\tilde{L}}{r^2}. \end{aligned} \tag{7}$$

B. Range of radial motion

Before solving for the actual trajectory, we consider the legal range of radii and the implications for the global structure of the trajectory. It is required that the square root in Eq. (7) have a semipositive argument:

$$-1 + \frac{\tilde{E}^2}{1-2M/r} - \frac{\tilde{L}^2}{r^2} \geq 0. \tag{8}$$

At the radii where equality holds, the particle's radial velocity is zero; this occurs at the *periastron* or *apastron* (closest or farthest points of the orbit). If we define the *effective potential* in analogy to the Newtonian central force problem:

$$\tilde{V}^2(r) = \left(1-2\frac{M}{r}\right)\left(1+\frac{\tilde{L}^2}{r^2}\right) = 1 - 2\frac{M}{r} + \frac{\tilde{L}^2}{r^2} - 2\frac{M\tilde{L}^2}{r^3}, \tag{9}$$

then our requirement is that

$$\tilde{E}^2 \geq \tilde{V}^2(r). \tag{10}$$

In our case, for particles moving forward in time we have $\tilde{E} > 0$, so really we could write the requirement as

$$\tilde{E} \geq \tilde{V}(r). \tag{11}$$

Therefore the range of radial motion is set by the effective potential and the energy. At large distances from the star, in the Newtonian limit, the effective potential becomes

$$\tilde{V}(r) \approx V_{\text{Kepler}}(r) = 1 - \frac{M}{r} + \frac{\tilde{L}^2}{2r^2}, \tag{12}$$

and the usual rules hold: there is a minimum of the potential (except for radial orbits); particles oscillating in this minimum orbit the star between fixed periastron and apastron distances; and if $\tilde{E} \geq 1$ the orbit is unbound (parabolic or hyperbolic).

What about in GR? Here instead of the Newtonian limit of $\tilde{V}(r)$ increasing without bound as $r \to 0$, the effective potential must turn around and reach zero at $r = 2M$. It can't wiggle around too many times though: $\tilde{V}^2(r)$ is a cubic polynomial in $1/r$, so one distinguishes three possibilities:

- The effective potential may rise from $\tilde{V}(2M) = 0$ to a maximum, then decline to a minimum and then rise again to $\tilde{V}(\infty) = 1$.
- The effective potential may rise monotonically.
- A limiting case where the effective potential has a simultaneous stationary and inflection point, i.e. it is monotonic but there is a point r_i where $\tilde{V}'(r_i) = 0$ and $\tilde{V}''(r_i) = 0$.

We may distinguish these cases by searching for the extremal points where $d\tilde{V}/dr = 0$. That is equivalent to asking for

$$0 = \frac{1}{2}\frac{d[\tilde{V}^2(r)]}{d[1/r]} = \frac{M}{r^2} - \frac{\tilde{L}^2}{r^3} + 3\frac{M\tilde{L}^2}{r^4}. \tag{13}$$

This is a quadratic equation in $1/r$, with solutions

$$\frac{1}{r} = \frac{\tilde{L} \mp \sqrt{\tilde{L}^2 - 12M^2}}{6M\tilde{L}}. \tag{14}$$

Here the + sign corresponds to the larger value of r (the minimum of $\tilde{V}$) and the − sign to the smaller value of r (the maximum of $\tilde{V}$).

The three cases are distinguished by how $\tilde{L}$ compares to the critical value $\tilde{L}_c = 2\sqrt{3}\,M$:

- If $\tilde{L} > \tilde{L}_c$, then the effective potential contains a minimum and maximum.
- If $\tilde{L} < \tilde{L}_c$, then the effective potential is monotonic. In this case there are no stable bound orbits!
- If $\tilde{L} = \tilde{L}_c$, then the effective potential is instantaneously flat at $r = r_+ = r_- = 6M$.

C. Circular orbits

We next consider the simplest orbits around a spherical object – the circular orbits. These lie at a minimum of the effective potential. They may be parameterized by either their specific angular momentum or their radius coordinate r; we take the latter choice. In order for the orbit to be circular, we have $d\tilde{V}/dr = 0$, or equivalently Eq. (13) is satisfied. Thus algebraically we find

$$\tilde{L} = \sqrt{\frac{Mr^2}{r - 3M}}. \tag{15}$$

The corresponding specific energy of the circular orbit is

$$\tilde{E} = \tilde{V}(r) = \sqrt{\left(1 - 2\frac{M}{r}\right)\left(1 + \frac{Mr^2/(r-3M)}{r^2}\right)} = \frac{r - 2M}{\sqrt{r(r-3M)}}. \tag{16}$$

At large radii these reduce to the Keplerian expressions $\sqrt{Mr}$ and $1 - M/(2r)$, respectively.

The orbital period of the test particle, as seen by an observer far from the hole, is the (Schwarzschild coordinate) time required for the particle to complete one lap, i.e.

$$T = \frac{2\pi}{\Omega} = \frac{2\pi}{d\phi/dt}. \tag{17}$$

This orbital frequency is

$$\Omega = \frac{u^\phi}{u^t} = \frac{\tilde{L}/r^2}{\tilde{E}/(1-2M/r)} = \frac{\sqrt{M/(r-3M)}/r}{r/\sqrt{r(r-3M)}} = \frac{M^{1/2}}{r^{3/2}}. \tag{18}$$

This is actually identical to the Keplerian result! The agreement is however not physically meaningful since r represents merely a radial coordinate.

The actual behavior of circular orbits at small radii is however very different from the Keplerian result. Recall that in order to have a stable circular orbit, we must be at a minimum of the effective potential. However we see from

Eq. (14) that the minima of $\tilde{V}$ are always at $r > 6M$ and the maxima are at $r < 6M$. Therefore, we see that there is an *innermost stable circular orbit* (ISCO) at $r = 6M$. Its specific energy and angular momentum are

$$\tilde{E}(r_{\rm ISCO}) = \frac{2\sqrt{2}}{3} \quad \text{and} \quad \tilde{L}(r_{\rm ISCO}) = 2\sqrt{3}. \tag{19}$$

Outside the ISCO, one has the usual behavior: small perturbations on the orbit lead to small radial oscillations, and moreover the inner orbits have lower energy and angular momentum than the outer orbits, i.e. $d\tilde{L}/dr > 0$ and $d\tilde{E}/dr > 0$.

Inside the ISCO, strange behaviors occur. In addition to such orbits being unstable to radial perturbations, and hence of little direct astrophysical interest, one has $d\tilde{L}/dr < 0$ and $d\tilde{E}/dr < 0$. In fact at $r = 4M$, the energy of the circular orbit reaches 1: inside this radius, a small perturbation on a circular orbit leads to not just an instability, but (if the perturbation is outward) the test particle escaping to ∞! This is called the *marginally bound* orbit. Finally at $r = 3M$, the specific energy and angular momentum become infinite. Such an orbit is allowed only by setting $\mu = 0$ and hence the $r = 3M$ radius is called the *photon sphere*. At the photon sphere, light can orbit the star (assuming $R < 3M$ or it has become a black hole), but of course such trajectories are unstable. At $r < 3M$ there are no circular orbits at all.

The ISCO is very important in the study of astrophysical accretion disks around black holes, since for thin, zero-pressure disks the gas parcels are on near-circular orbits and the ISCO then represents the inner edge.

IV. APPLICATION: PERIHELION PRECESSION

We now turn our attention to more general motions of massive test particles, which have both radial and angular motion. An obvious question is whether the orbit closes: does a particle follow the same closed loop around a star in GR on orbit after orbit, as occurs with Kepler's ellipses? To answer this question, we need to know the angular advance of the particle during a radial cycle. That is, we find its trajectory $\phi(r)$ by using

$$\frac{d\phi}{dr} = \frac{u^\phi}{u^r}, \tag{20}$$

and then integrate over a radial cycle:

$$\Delta\phi = \oint \frac{d\phi}{dr}\, dr. \tag{21}$$

Explicitly, we have

$$\Delta\phi = \oint \pm \frac{\tilde{L}/r^2}{\sqrt{\tilde{E}^2 - (1 - 2M/r)(1 + \tilde{L}^2/r^2)}}\, dr. \tag{22}$$

This equation is simplified if we substitute $\upsilon = 1/r$, in which case

$$\Delta\phi = \oint \pm \frac{\tilde{L}}{\sqrt{\tilde{E}^2 - (1 - 2M\upsilon)(1 + \tilde{L}^2\upsilon^2)}}\, d\upsilon. \tag{23}$$

The exact result for Eq. (23) contains elliptic integrals and we will not solve it in class. We will however solve the case where the radial perturbations are small: that is, where we are near the minimum $\upsilon_0 = 1/r_0$ of the effective potential (near a circular orbit). Then the specific angular momentum is related to the minimum via the usual relation

$$\tilde{L}^2 = \frac{Mr_0^2}{r_0 - 3M} = \frac{M}{\upsilon_0(1 - 3M\upsilon_0)}. \tag{24}$$

The specific energy now must exceed $\tilde{V}(r_0)$ in order for the particle to oscillate around the minimum: we suppose it does so by an amount ϵ:

$$\tilde{E}^2 = \tilde{V}^2(r_0) + \epsilon = \frac{(1 - 2M\upsilon_0)^2}{1 - 3M\upsilon_0} + \epsilon. \tag{25}$$

We now take $\upsilon = \upsilon_0 + \delta$, and consider the argument of the square root in Eq. (23). We recall that since

$$\tilde{V}^2(\upsilon) = (1 - 2M\upsilon)(1 + \tilde{L}^2\upsilon^2) \tag{26}$$

is a cubic and we are expanding around the minimum, the Taylor expansion is

$$\tilde{V}^2(\upsilon) = \tilde{V}^2(\upsilon_0) + \tilde{L}^2(1 - 6M\upsilon_0)\delta^2 - 2M\tilde{L}^2\delta^3. \tag{27}$$

For small radial oscillations we neglect the cubic term in the potential, leading to the following simplification of Eq. (23):

$$\Delta\phi = \oint \pm \frac{\tilde{L}}{\sqrt{\epsilon - \tilde{L}^2(1 - 6M\upsilon_0)\delta^2}}\, d\delta. \tag{28}$$

Now to simplify this further we need to understand the range of the radial perturbation variable δ. It oscillates between the limits at which $u^r = 0$, i.e. where the argument of the square root is zero:

$$|\delta| < \delta_{\max} = \sqrt{\frac{\epsilon}{\tilde{L}^2(1 - 6M\upsilon_0)}}. \tag{29}$$

We thus set

$$\delta = \delta_{\max} \cos\zeta, \tag{30}$$

so that a full cycle of radial oscillation corresponds to incrementing ζ by 2π. Correspondingly, we choose the + sign in Eq. (28) for half of the oscillation and the − sign for the other half. We then have

$$\Delta\phi = \int_0^{2\pi} \mp \frac{\tilde{L}}{\sqrt{\epsilon - \epsilon\cos^2\zeta}}\, \delta_{\max} \sin\zeta \, d\zeta = \frac{\tilde{L}\delta_{\max}}{\sqrt{\epsilon}} \int_0^{2\pi} d\zeta = 2\pi \frac{\tilde{L}\delta_{\max}}{\sqrt{\epsilon}}. \tag{31}$$

Using our expression for $\delta_{\max}$, this leads to

$$\Delta\phi = \frac{2\pi}{\sqrt{1 - 6M\upsilon_0}} = \frac{2\pi}{\sqrt{1 - 6M/r_0}}. \tag{32}$$

The orbit of the test particle very nearly closes if r_0 is large. However there is an excess advance of the perihelion per orbit given by:

$$\Psi = \Delta\phi - 2\pi = 2\pi \left(\frac{1}{\sqrt{1 - 6M/r_0}} - 1 \right) \approx \frac{6\pi M}{r_0}. \tag{33}$$

This is positive, indicating that the perihelion moves forward. In the case of the innermost planet Mercury around the Sun, we have

$$\Psi = \frac{6\pi G M_\odot}{c^2 r_0} = \frac{6\pi (6.672 \times 10^{-8}\,\mathrm{cm^3\,g^{-1}\,s^{-2}})(2 \times 10^{33}\,\mathrm{g})}{(3 \times 10^{10}\,\mathrm{cm\,s^{-1}})^2 (5.8 \times 10^{12}\,\mathrm{cm})} = 4.8 \times 10^{-7}\,\mathrm{rad} = 0.10\,\mathrm{arcsec}. \tag{34}$$

This is a tiny angle, but it builds up each orbit (every 88 Earth-days in the case of Mercury). Over the course of a century, Mercury's orbit thus advances by 40 arcsec due to relativistic effects. Thanks to Mercury's high eccentricity ($e = 0.2$), this angle was large enough for astronomers to observe even in the 19th century, thus making it the first manifestation of GR to be discovered, and the only one known prior to the formulation of the theory.

[There is a contribution a factor of a few larger due to the gravitational perturbations from the other planets; but since the masses of e.g. Venus, Earth, and Jupiter are known, these could be removed and the residuals discovered.]