Lecture XVIII: Spherical stars – I. The radial structure equations

Christopher M. Hirata
Caltech M/C 350-17, Pasadena CA 91125, USA[*]
(Dated: January 13, 2012)

I. OVERVIEW

We now consider the structure of spherical stars in GR. We consider the basic properties and the system of equations in this lecture. The peculiar properties of the solutions (different from Newtonian gravity) are considered here, and in the next lecture we look at specific examples.

The reading for this lecture is:

- MTW Ch. 23.

II. PRELIMINARIES

We begin with the metric for a spherical system, derived in the last lecture,

$$ds^2 = -e^{2\Phi}dt^2 + e^{2\Lambda}dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2), \tag{1}$$

where $\Phi(r)$ and $\Lambda(r)$ are arbitrary functions of r. Our goal is to relate these functions to the density $\rho(r)$ and pressure $p(r)$ for a star in equilibrium. For a parcel of fluid in the star, if it is "at rest" in the sense that its motion is parallel to the time-translation Killing field, the 4-velocity is

$$u^\alpha = (e^{-\Phi}, 0, 0, 0) = e^{-\Phi}\xi^\alpha, \tag{2}$$

where $\boldsymbol{\xi}$ is the timelike Killing field.

The stress-energy tensor is

$$T^{\mu\nu} = \rho u^\mu u^\nu + p(g^{\mu\nu} + u^\mu u^\nu), \tag{3}$$

or in this case:

$$T^{tt} = \rho e^{-2\Phi}, \quad T^{rr} = pe^{-2\Lambda}, \quad T^{\theta\theta} = \frac{p}{r^2}, \quad \text{and} \quad T^{\phi\phi} = \frac{p}{r^2\sin^2\theta}, \tag{4}$$

with other components zero.

In Newtonian gravity, a spherical star is specified uniquely by $\rho(r)$. We therefore expect that in GR there will be one free function as well: out of the 4 functions Φ, Λ, p, and ρ, we will have only 3 independent equations. This will turn out to indeed be the case.

III. THE TOLMAN-OPPENHEIMER-VOLKOFF EQUATIONS

We now begin our consideration of the structural equations of the star. We are searching for three equations, which come from some combination of energy-momentum conservation and the Einstein equations.

---

[*]Electronic address: `chirata@tapir.caltech.edu`

A. Energy-momentum conservation

Our first consideration is energy-momentum conservation: using the equation $\boldsymbol{u} = e^{-\Phi}\boldsymbol{\xi}$, we find that the divergence of Eq. (3) becomes

$$\begin{aligned} T^{\mu\nu}{}_{;\nu} &= \nabla_\nu[\rho u^\mu u^\nu + p(g^{\mu\nu} + u^\mu u^\nu)] \\ &= \nabla_\nu[e^{-2\Phi}(\rho+p)\xi^\mu\xi^\nu + pg^{\mu\nu}] \\ &= [e^{-2\Phi}(\rho+p)]_{;\nu}\xi^\nu + e^{-2\Phi}(\rho+p)\xi^\mu{}_{;\nu}\xi^\nu + e^{-2\Phi}(\rho+p)\xi^\mu\xi^\nu{}_{;\nu} + p^{;\mu}. \end{aligned} \tag{5}$$

In this equation, the first term vanishes since $e^{-2\Phi}(\rho+p)$ is a scalar that depends only on r, so its covariant derivative has only an r component and thus gives zero when contracted with ξ^ν. The third term is also zero because $\boldsymbol{\xi}$ is a Killing field and thus $\xi^\nu{}_{;\nu} = g^{\mu\nu}\xi_{(\mu;\nu)} = 0$. We therefore have

$$0 = e^{-2\Phi}(\rho+p)\xi_{\mu;\nu}\xi^\nu + p_{;\mu}. \tag{6}$$

Even this equation simplifies since for $\boldsymbol{\xi}$ a Killing field,

$$\xi_{\mu;\nu}\xi^\nu = -\xi_{\nu;\mu}\xi^\nu = -\frac{1}{2}(\xi^\nu\xi_\nu)_{;\mu} = -\frac{1}{2}(-e^{2\Phi})_{,\mu}, \tag{7}$$

and hence

$$0 = \frac{1}{2}e^{-2\Phi}(\rho+p)(e^{2\Phi})_{,\mu} + p_{,\mu}. \tag{8}$$

Only the r-component of this equation is nontrivial, and it gives us our first equation:

$$\frac{dp}{dr} = -(\rho+p)\frac{d\Phi}{dr}. \tag{9}$$

This is the familiar equation of hydrostatic equilibrium: the pressure gradient balances the "acceleration due to gravity" (in Newtonian language).

Note that in GR both the density *and* pressure appear on the right-hand side of the equation. Under ordinary conditions we don't care: for example, in the core of the Sun, the pressure is thermal and

$$\frac{p}{\rho} = \frac{kT}{\mu} = \frac{1\ \text{keV}}{1\ \text{GeV}} \sim 10^{-6}. \tag{10}$$

But in a neutron star (where the pressure comes from nuclear forces) p/ρ is typically of order 10^{-1}. GR is required to compute its structure accurately.

B. The curvature tensor

For the remainder of the problem, there is no simple substitute for calculating the curvature tensor. Fortunately, the Christoffel symbols are functions only of r and θ, and most vanish by symmetry. The nonzero ones are:

$$\begin{aligned} \Gamma^t_{tr} = \Gamma^t_{rt} &= \frac{d\Phi}{dr}, \\ \Gamma^r_{tt} &= e^{2\Phi-2\Lambda}\frac{d\Phi}{dr}, \\ \Gamma^r_{rr} &= \frac{d\Lambda}{dr}, \\ \Gamma^r_{\theta\theta} &= -re^{-2\Lambda}, \\ \Gamma^r_{\phi\phi} &= -re^{-2\Lambda}\sin^2\theta, \\ \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} &= \frac{1}{r}, \\ \Gamma^\theta_{\phi\phi} &= -\sin\theta\cos\theta, \\ \Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} &= \frac{1}{r}, \quad \text{and} \\ \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} &= \cot\theta. \end{aligned} \tag{11}$$

The intermediate sums are

$$\Gamma^\alpha_{\alpha t} = 0, \quad \Gamma^\alpha_{\alpha r} = \frac{d\Phi}{dr} + \frac{d\Lambda}{dr} + \frac{2}{r}, \quad \Gamma^\alpha_{\alpha\theta} = \cot\theta. \tag{12}$$

The implied Ricci tensor is

$$R_{\mu\nu} = \Gamma^\gamma_{\mu\nu,\gamma} + \Gamma^\gamma_{\delta\gamma}\Gamma^\delta_{\mu\nu} - \Gamma^\gamma_{\mu\gamma,\nu} - \Gamma^\gamma_{\delta\nu}\Gamma^\delta_{\mu\gamma}; \tag{13}$$

the components that are not trivially zero are

$$\begin{aligned}
R_{tt} &= \frac{d}{dr}\left(e^{2\Phi-2\Lambda}\frac{d\Phi}{dr}\right) + \left(\frac{d\Phi}{dr} + \frac{d\Lambda}{dr} + \frac{2}{r}\right)e^{2\Phi-2\Lambda}\frac{d\Phi}{dr} - 2e^{2\Phi-2\Lambda}\left(\frac{d\Phi}{dr}\right)^2, \\
R_{rr} &= \frac{d^2\Lambda}{dr^2} + \left(\frac{d\Phi}{dr} + \frac{d\Lambda}{dr} + \frac{2}{r}\right)\frac{d\Lambda}{dr} - \frac{d}{dr}\left(\frac{d\Phi}{dr} + \frac{d\Lambda}{dr} + \frac{2}{r}\right) - \left(\frac{d\Phi}{dr}\right)^2 - \left(\frac{d\Lambda}{dr}\right)^2 - \frac{2}{r^2}, \\
R_{\theta\theta} &= -\frac{d}{dr}(re^{-2\Lambda}) - \left(\frac{d\Phi}{dr} + \frac{d\Lambda}{dr} + \frac{2}{r}\right)re^{-2\Lambda} + \csc^2\theta + 2e^{-2\Lambda} - \cot^2\theta, \quad \text{and} \\
R_{\phi\phi} &= \sin^2\theta\, R_{\theta\theta}.
\end{aligned} \tag{14}$$

It is advantageous to form a local orthonormal set of basis vectors,

$$\boldsymbol{e}_{\hat t} = \boldsymbol{u} = e^{-\Phi}\boldsymbol{e}_t, \quad \boldsymbol{e}_{\hat r} = e^{-\Lambda}\boldsymbol{e}_r, \quad \boldsymbol{e}_{\hat\theta} = \frac{1}{r}\boldsymbol{e}_\theta, \quad \text{and} \quad \boldsymbol{e}_{\hat\phi} = \frac{1}{r\sin\theta}\boldsymbol{e}_\phi. \tag{15}$$

In this basis, the Ricci tensor becomes

$$\begin{aligned}
R_{\hat t\hat t} &= e^{-2\Lambda}\left[\frac{d^2\Phi}{dr^2} + \left(\frac{d\Phi}{dr}\right)^2 - \frac{d\Lambda}{dr}\frac{d\Phi}{dr} + \frac{2}{r}\frac{d\Phi}{dr}\right], \\
R_{\hat r\hat r} &= e^{-2\Lambda}\left[\frac{d\Phi}{dr}\frac{d\Lambda}{dr} + \frac{2}{r}\frac{d\Lambda}{dr} - \frac{d^2\Phi}{dr^2} - \left(\frac{d\Phi}{dr}\right)^2\right], \quad \text{and} \\
R_{\hat\theta\hat\theta} = R_{\hat\phi\hat\phi} &= \frac{1}{r^2}e^{-2\Lambda}\left[e^{2\Lambda} - 1 + r\left(\frac{d\Lambda}{dr} - \frac{d\Phi}{dr}\right)\right].
\end{aligned} \tag{16}$$

C. The density equation

We are now ready to consider the Einstein tensor: first, we consider the density equation:

$$8\pi\rho = G_{\hat t\hat t} = R_{\hat t\hat t} + \frac{1}{2}R = \frac{1}{2}(R_{\hat t\hat t} + R_{\hat r\hat r} + R_{\hat\theta\hat\theta} + R_{\hat\phi\hat\phi}) = \frac{2}{r}e^{-2\Lambda}\frac{d\Lambda}{dr} + \frac{1}{r^2}(1 - e^{-2\Lambda}). \tag{17}$$

This does not involve Φ. There is in fact a deep reason for this: the equations for energy and momentum density in GR are "constraint equations," analogous to Gauss's law for the charge density in electrodynamics, and in time-symmetric ($t \rightarrow -t$) situations the constraint includes only the intrinsic spatial curvature; but we will leave these matters for the third term. But for the moment, our emphasis is on finding the equations of stellar structure, and so we will simply use this result to obtain the relation of Λ to ρ. We can see that

$$\frac{d}{dr}\left[r(1 - e^{-2\Lambda})\right] = 1 - e^{-2\Lambda} - 2re^{-2\Lambda}\frac{d\Lambda}{dr} = 8\pi r^2\rho. \tag{18}$$

We can then define the function

$$m(r) \equiv \frac{1}{2}r(1 - e^{-2\Lambda}) \quad \leftrightarrow \quad e^{-2\Lambda} = 1 - 2\frac{m}{r}, \tag{19}$$

which is given by

$$m(r) = 4\pi\int r^2\rho(r)\,dr. \tag{20}$$

So far $m(r)$ is just a symbol, but as you can probably guess it is going to be the relativistic version of "enclosed mass." (We will have to prove later that $m(r \rightarrow \infty)$ can be identified with the total gravitational mass of an isolated system, which we previously defined.)

We have also not specified the integration constant: if the star has a well-behaved center (like the Sun) then $m(r = 0) = 0$. If $m(r = 0) > 0$ then at some point one has $2m/r = 1$, and one might guess that strange things happen at that radius – Λ becomes infinite! This case will correspond to a spherical black hole. It is not our concern yet.

D. The transverse pressure equation

The next equation we consider is the transverse pressure equation. We must have

$$R_{\hat{\theta}\hat{\theta}} = G_{\hat{\theta}\hat{\theta}} - \frac{1}{2}G = 8\pi p - \frac{1}{2}(8\pi)(-\rho + 3p) = 4\pi(\rho - p). \tag{21}$$

From Eq. (16) we then find

$$4\pi(\rho - p) = \frac{1}{r^2}e^{-2\Lambda}\left[e^{2\Lambda} - 1 + r\left(\frac{d\Lambda}{dr} - \frac{d\Phi}{dr}\right)\right]. \tag{22}$$

We can substitute m in place of Λ. Noting that

$$\frac{d\Lambda}{dr} = \frac{d}{dr}\left[-\frac{1}{2}\ln\left(1 - 2\frac{m}{r}\right)\right] = -\frac{1}{2}\frac{2m/r^2 - 2r^{-1}\,dm/dr}{1 - 2m/r} = \frac{dm/dr - m/r}{r - 2m} = \frac{4\pi r^2\rho - m/r}{r - 2m}, \tag{23}$$

this yields

$$4\pi(\rho - p) = \frac{1}{r^2}\left(1 - 2\frac{m}{r}\right)\left[\frac{1}{1 - 2m/r} - 1 + r\frac{4\pi r^2\rho - m/r}{r - 2m} - r\frac{d\Phi}{dr}\right]. \tag{24}$$

We may simplify this algebraically. The term involving ρ on the right-hand side can be extracted to cancel the $4\pi\rho$ on the left. Then:

$$-4\pi p = \frac{m}{r^3} - \frac{1}{r}\left(1 - 2\frac{m}{r}\right)\frac{d\Phi}{dr}. \tag{25}$$

We therefore have a single differential equation for Φ:

$$\frac{d\Phi}{dr} = \frac{m + 4\pi r^3 p}{r(r - 2m)}. \tag{26}$$

E. Other components

The remaining 8 components of the Einstein equations contain no additional information. Of these:

- 3 of them ($G_{\hat{t}\hat{r}}$, $G_{\hat{t}\hat{\theta}}$, and $G_{\hat{t}\hat{\phi}}$) vanish by time reversal symmetry.
- 4 of them ($G_{\hat{r}\hat{\theta}}$, $G_{\hat{r}\hat{\phi}}$, $G_{\hat{\theta}\hat{\phi}}$, and $G_{\hat{\theta}\hat{\theta}} - G_{\hat{\phi}\hat{\phi}}$) vanish by spherical symmetry.
- The last equation is redundant with the continuity equation $T^{\hat{r}\alpha}{}_{;\alpha} = 0$ due to the Bianchi identity.

F. Summary

We now summarize the system of equations we have derived for relativistic stars. The metric is

$$ds^2 = -e^{2\Phi}dt^2 + \frac{dr^2}{1 - 2m/r} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{27}$$

where Φ and m are functions of r. They are related to the pressure and density via

$$\frac{dm}{dr} = 4\pi r^2\rho \quad \text{and} \quad \frac{d\Phi}{dr} = \frac{m + 4\pi r^3 p}{r(r - 2m)}, \tag{28}$$

and the equation of hydrostatic equilibrium, Eq. (9) can be written using the Einstein equation for $d\Phi/dr$ as

$$\frac{dp}{dr} = -\frac{(m + 4\pi r^3 p)(\rho + p)}{r(r - 2m)}. \tag{29}$$

These are known as the *Tolman-Oppenheimer-Volkoff equations* (or "TOV" equations).

IV. EXTERIOR SOLUTION

For a star with a finite concentration of mass, we have outside the star's radius (at $r > R$) that m is constant (call it M). Then Eq. (28) gives the result for Φ,

$$\Phi = \int \frac{d\Phi}{dr} dr = \int \frac{M}{r(r-2M)} dr = \frac{1}{2}\int \left(\frac{1}{r-2M} - \frac{1}{r}\right) dr = \frac{1}{2}\ln\frac{r-2M}{r} + C = \frac{1}{2}\ln\left(1 - 2\frac{M}{r}\right) + C, \tag{30}$$

where C is a constant. In fact C is physically irrelevant: by a rescaling $t \to e^{-C}t$, we can eliminate it. Doing so, we find the metric,

$$ds^2 = -\left(1 - 2\frac{M}{r}\right) dt^2 - \frac{dr^2}{1 - 2M/r} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{31}$$

This is called the *Schwarzschild metric* and it applies to the spacetime outside of any spherical star. (Actually we have just proved this here for time-independent stars, but it is true in general.)

By examining the $\mathcal{O}(1/r)$ terms in the metric we can see that M corresponds to the gravitational mass of the star. Assuming a well-behaved core, this is

$$M = \int_0^R 4\pi r^2 \rho(r)\, dr, \tag{32}$$

which despite appearances is **not** the volume integral of the density since the 3-volume element is not $4\pi r^2\, dr$.

Note that if a star evolves from one state to another, then the conservation of effective stress-energy (first term!) then tells us that

$$M_{\rm new} = M_{\rm old} - \int \left[\oint T^{{\rm eff}\ 0i} n_i\, d^2x\right] dt, \tag{33}$$

where the effective energy flux $T^{{\rm eff}\ 0i}$ includes everything the star emits – winds, light, neutrinos, and (if applicable) gravitational radiation.

V. EXAMPLES

It is now of interest to examine the solutions of the TOV system. We consider first the nonrelativistic limit, and then examine some types of relativistic stars.

A. Nonrelativistic star

In the nonrelativistic limit, $p \ll \rho$, $m \ll r$, and $r^3 p \ll m$. Then the TOV system becomes

$$\frac{dm}{dr} = 4\pi r^2 \rho, \quad \frac{d\Phi}{dr} = \frac{m}{r^2}, \quad \text{and} \quad \frac{dp}{dr} = -\frac{m\rho}{r^2}. \tag{34}$$

These are the ordinary equations of stellar structure.

B. Uniform density star

The simplest fully relativistic solution concerns a star of constant density ρ_0 out to a maximum radius R, with vacuum outside. This example also illustrates many of the bizarre features of relativistic stars.

For this case, we can immediately see that inside the star, $m = \frac{4}{3}\pi\rho_0 r^3$. Then Eq. (29) gives a single ODE for the pressure:

$$\frac{dp}{dr} = -\frac{4\pi r(\rho_0/3 + p)(\rho_0 + p)}{1 - 8\pi\rho_0 r^2/3}. \tag{35}$$

The equation easily separates:

$$\int \frac{dp}{(\rho_0/3+p)(\rho_0+p)} = -\int \frac{4\pi r\,dr}{1-8\pi\rho_0 r^2/3}. \tag{36}$$

Performing the integrals on both sides gives

$$\frac{3}{2\rho_0}\ln\frac{\rho_0/3+p}{\rho_0+p} = \frac{3}{4\rho_0}\ln\left(1-\frac{8}{3}\pi\rho_0 r^2\right) + \text{const}. \tag{37}$$

Rearranging gives

$$\frac{\rho_0/3+p}{\rho_0+p} = C\sqrt{1-\frac{8}{3}\pi\rho_0 r^2}, \tag{38}$$

where C is a constant. The value of C can be obtained by setting the pressure to zero at the surface, yielding

$$\frac{\rho_0+3p}{\rho_0+p} = \sqrt{\frac{1-8\pi\rho_0 r^2/3}{1-8\pi\rho_0 R^2/3}}, \tag{39}$$

and then

$$p = \rho_0\frac{\sqrt{(1-8\pi\rho_0 r^2/3)/(1-8\pi\rho_0 R^2/3)}-1}{3-\sqrt{(1-8\pi\rho_0 r^2/3)/(1-8\pi\rho_0 R^2/3)}}. \tag{40}$$

Of particular interest to us is the central pressure,

$$p_c = \rho_0\frac{1/\sqrt{1-8\pi\rho_0 R^2/3}-1}{3-1/\sqrt{1-8\pi\rho_0 R^2/3}}, \tag{41}$$

or using $8\pi\rho_0 R^2/3 = 2M/R$:

$$p_c = \rho_0\frac{1/\sqrt{1-2M/R}-1}{3-1/\sqrt{1-2M/R}} = \frac{3M}{4\pi R^3}\,\frac{1-\sqrt{1-2M/R}}{3\sqrt{1-2M/R}-1}. \tag{42}$$

Again, in the limit of $M/R \ll 1$ this is well behaved: the Taylor expansion of the square roots gives

$$p_c \to \frac{3M^2}{8\pi R^4}, \tag{43}$$

which is the Newtonian result. However, as M/R grows, so does the central pressure; it is greater than one would predict from Eq. (44). In particular, we find $p_c \to \infty$ when

$$\frac{M}{R} = \frac{4}{9}. \tag{44}$$

Thus general relativity has a maximum amount of mass that can be packed into a uniform-density object. It can be shown that the inequality

$$\frac{M}{R} < \frac{4}{9} \tag{45}$$

remains valid for any object whose density declines outward. This type of limit is unique to GR.

One may alternatively write this as the maximum gravitational redshift for an equilibrium star: the gravitational redshift at the surface, obtained from conservation of p_t for a photon, is

$$z_{\rm grav} \equiv \frac{\lambda(r=\infty)}{\lambda(r=R)} - 1 = \frac{1}{\sqrt{1-2M/R}} - 1 < 2. \tag{46}$$