Lecture XVII: Spherical stars – I. The metric structure

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I. OVERVIEW

We now begin our consideration of spherically symmetric spacetimes, of which relativistic stars and Schwarzschild black holes will be our principal examples. First on our agenda will be spherical stars. We consider the metric structure in this lecture, and then the equations of structure in the next lecture.

The reading for this lecture is:

- MTW Ch. 24.

II. SPHERICAL SYMMETRY

We begin our investigation with the case of spherical symmetry. Recall that this is defined by the existence of Killing vector fields $\{\boldsymbol{J}_i\}_{i=1}^3$ satisfying

$$[\boldsymbol{J}_i, \boldsymbol{J}_j] = -\epsilon_{ijk}\boldsymbol{J}_k. \tag{1}$$

[Recall that here the $_i$ index denotes which vector, *not* a component.]

Our first task is to learn something about the vector fields $\boldsymbol{J}_i$. We build up this knowledge in stages.

A. Preliminary considerations

We begin by noting that for any Killing vector field $\boldsymbol{\xi}$, it is possible to define a finite symmetry operation $G_{\boldsymbol{\xi}}$: $\mathcal{M} \rightarrow \mathcal{M}$ by taking the differential equation

$$\frac{d\mathcal{A}(s)}{ds} = \boldsymbol{\xi}[\mathcal{A}(s)], \tag{2}$$

initializing it at $\mathcal{A}(s_{\rm initial} = 0) = \mathcal{P}$, and setting $G_{\boldsymbol{\xi}}(\mathcal{P}) = \mathcal{A}(s_{\rm final} = 1)$. That this operation is a symmetry of the spacetime follows directly from differentiating the metric tensor $g_{\mu\nu}$ on the final manifold with respect to s, and obtaining $\mathcal{L}_{\boldsymbol{\xi}} g_{\mu\nu} = 0$. For a trivial example, we see that if $\boldsymbol{\xi}$ is the time-translation Killing field of Minkowski space, then the finite symmetry operation $G_{c\boldsymbol{\xi}}$ increments the time by c; hence it is appropriate to consider this a finite symmetry operation.

We now consider a point $\mathcal{P}$ in a spherically symmetric spacetime and consider the submanifold $\mathcal{S}(\mathcal{P}) \subset \mathcal{M}$ that is reachable by finite symmetry operations using the $\{\boldsymbol{J}_i\}$. This submanifold is the set of points that are "equivalent" to $\mathcal{P}$ because of the symmetry. We expect it to be a sphere, although there are some subtleties.

It is not immediately obvious what is the dimension of this submanifold (called an *orbit*) for a generic point $\mathcal{P}$. Clearly the vectors $\{\boldsymbol{J}_i\}_{i=1}^3$ must be tangent to this submanifold, since for any of the $\boldsymbol{J}_i$ the points $G_{c\boldsymbol{J}_i}(\mathcal{P})$ form a curve in $S(\mathcal{P})$ and the tangent vector to that curve is $\boldsymbol{J}_i$. So one may immediately conclude that:

- Since at least one of the $\boldsymbol{J}_i$ must be nonzero at a generic point (remember it is a basis Killing field), then for a generic point $\mathcal{S}(\mathcal{P})$ has dimension at least 1.

- But the dimension of $\mathcal{S}(\mathcal{P})$ cannot generically be 1 since then we would have to have $\boldsymbol{J}_1$, $\boldsymbol{J}_2$, and $\boldsymbol{J}_3$ all parallel – they must be tangent to the 1D curve $\mathcal{S}(\mathcal{P})$. Then one can choose one of these (say $\boldsymbol{J}_3$) to be nonzero somewhere, and then construct a coordinate system in which $\boldsymbol{J}_3$ has contravariant components $(J_3)^\alpha = (0, 0, 0, 1)$. We then

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have $(J_1)^\alpha = (0,0,0,a)$ and $(J_2)^\alpha = (0,0,0,b)$ for some functions a and b. The three commutation relations Eq. (1) then imply

$$ab_{,3} - ba_{,3} = -1, \quad - b_{,3} = -a, \quad \text{and} \quad a_{,3} = -b. \tag{3}$$

This implies $a^2 + b^2 = -1$, so not possible for real fields.

Therefore the generic dimension of $\mathcal{S}(\mathcal{P})$ is at least 2. We will proceed on this assumption for now. Cases with dimension 3 or more are in fact more symmetrical spacetimes, and we will come back to them later; they can always be reduced to the form below by an appropriate choice of fields. [1]

B. Killing vectors for the case of orbits of dimension 2

If the orbit $\mathcal{S}(\mathcal{P})$ has dimension 2, then the vectors $\boldsymbol{J}_1$, $\boldsymbol{J}_2$, and $\boldsymbol{J}_3$ are in the 2-dimensional tangent space to $\mathcal{S}(\mathcal{P})$. Therefore they are linearly dependent, and at any point $\mathcal{P}$ there are numbers $\{c^i\}_{i=1}^3$ such that

$$c^i(\mathcal{P})\boldsymbol{J}_i(\mathcal{P}) = 0. \tag{4}$$

[Remember: here i is not an index, it denotes which number.] By normalizing the c's to have the sum of their squares equal to 1, we may write

$$\sin\theta\cos\phi\,\boldsymbol{J}_1 + \sin\theta\sin\phi\,\boldsymbol{J}_2 + \cos\theta\,\boldsymbol{J}_3 = 0. \tag{5}$$

Here θ and ϕ are functions of $\mathcal{P}$ that are defined up to an overall sign degeneracy in most places (i.e. one may take $\theta \to \pi - \theta$ and $\phi \to \phi + \pi$ and the above equation is still valid). We make a continuous choice of such functions and will treat them as two of our coordinates. It takes two more coordinates (let us call them x^0 and x^1) to describe which of the 2-dimensional manifolds a point is on. So our coordinate system is (x^0, x^1, θ, ϕ).

Conceptually, what we have done is to define a 2-dimensional shell $\mathcal{S}(\mathcal{P})$ indexed by (x^0, x^1), and constructed the angular coordinates (θ, ϕ) by which combination of rotations does not move that point (since it is the rotation operations that we have taken as fundamental).

Now since the $\boldsymbol{J}_i$ are tangent to the shell, we have $(J_i)^0 = (J_i)^1 = 0$ – i.e. if a point moves with velocity given by a $\boldsymbol{J}_i$, its x^0 and x^1 coordinates do not change.

It is possible at this stage to completely compute the components of all the $\boldsymbol{J}_i$. Consider e.g. $\boldsymbol{J}_3$. Recall that the Lie derivative of a coordinate is a component, e.g. $\mathcal{L}_{\boldsymbol{J}_3}\theta = (J_3)^\theta$; and that the commutation rule gives $\mathcal{L}_{\boldsymbol{J}_3}\boldsymbol{J}_1 = -\boldsymbol{J}_2$, etc. Then taking the $\mathcal{L}_{\boldsymbol{J}_3}$ derivative of Eq. (5) gives

$$[\cos\theta\cos\phi\,(J_3)^\theta - \sin\theta\sin\phi\,(J_3)^\phi]\boldsymbol{J}_1 - \sin\theta\cos\phi\,\boldsymbol{J}_2 + [\cos\theta\sin\phi\,(J_3)^\theta + \sin\theta\cos\phi\,(J_3)^\phi]\boldsymbol{J}_2 + \sin\theta\sin\phi\,\boldsymbol{J}_1 - \sin\theta\,(J_3)^\theta\,\boldsymbol{J}_3 = 0. \tag{6}$$

Now substituting for $\boldsymbol{J}_3$ using Eq. (5) gives

$$\begin{aligned}&[\cos\theta\cos\phi\,(J_3)^\theta - \sin\theta\sin\phi\,(J_3)^\phi + \sin\theta\sin\phi + \sin\theta\tan\theta\cos\phi\,(J_3)^\theta]\boldsymbol{J}_1\\ &+[\cos\theta\sin\phi\,(J_3)^\theta + \sin\theta\cos\phi\,(J_3)^\phi - \sin\theta\cos\phi + \sin\theta\tan\theta\sin\phi\,(J_3)^\theta]\boldsymbol{J}_2 = 0.\end{aligned} \tag{7}$$

Since $\boldsymbol{J}_1$ and $\boldsymbol{J}_2$ are generically linearly independent, both of the objects in brackets must be zero, which leads to $(J_3)^\theta = 0$ and $(J_3)^\phi = 1$. Thus $(J_3)^\alpha = (0,0,0,1)$.

The calculation for $\boldsymbol{J}_1$ and $\boldsymbol{J}_2$ is similar but messier; it leads to

$$(J_1)^\alpha = (0,0,-\sin\phi, -\cot\theta\cos\phi) \quad \text{and} \quad (J_2)^\alpha = (0,0,\cos\phi,-\cot\theta\sin\phi). \tag{8}$$

C. The metric tensor

We may now find the metric tensor components $g_{\mu\nu}$ in our chosen coordinate system. Clearly they are independent of ϕ. Moreover, at $\phi = 0$ we find that

$$0 = \mathcal{L}_{\boldsymbol{J}_2} g_{\mu\nu} = g_{\mu\nu,\theta} - g_{\mu\alpha}(J_2)^\alpha{}_{,\nu} - g_{\nu\alpha}(J_2)^\alpha{}_{,\mu}. \tag{9}$$

The $\mu\nu = 00$, 01, and 11 components immediately establish that g_{00}, g_{01}, and g_{11} are independent of θ.

At $\phi = 0$ the corresponding equation for $\mathcal{L}_{\boldsymbol{J}_1} g_{\mu\nu}$ reads

$$0 = \mathcal{L}_{\boldsymbol{J}_1} g_{\mu\nu} = -g_{\mu\alpha}(J_1)^\alpha{}_{,\nu} - g_{\nu\alpha}(J_1)^\alpha{}_{,\mu}. \tag{10}$$

The 03 component of this equation reads $0 = g_{02}$. The 02 component however reads $0 = -g_{03}\partial_\theta(-\cot\theta)$, and hence $g_{03} = 0$. Similar arguments show $g_{12} = g_{13} = 0$.

It remains to find g_{22}, g_{23}, and g_{33}. The 22 and 23 components of Eq. (10) yield

$$0 = -2g_{23}\partial_\theta(-\cot\theta) \quad \text{and} \quad 0 = g_{22} - g_{33}\partial_\theta(-\cot\theta). \tag{11}$$

The first establishes $g_{23} = 0$ and the second establishes $g_{22} = g_{33}\csc^2\theta$ or equivalently $g_{33} = g_{22}\sin^2\theta$. Moreover, the 22 component of Eq. (9) combined with our knowledge now that $g_{23} = 0$ gives $0 = g_{22,\theta}$. Thus g_{22} is a function only of x^0 and x^1; we denote it by r^2.

The overall metric is then of the form

$$ds^2 = g_{00}(dx^0)^2 + 2g_{01}dx^0dx^1 + g_{11}(dx^1)^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{12}$$

where g_{00}, g_{01}, g_{11}, and r^2 are functions of (x^0, x^1) only. This metric coincides with our intuitive notion of "spherical symmetry:" it is composed of many spheres of circumference $2\pi r$, labeled by (x^0, x^1) and stitched together in such a way that there is no preferred direction on the sphere.

[**Note**: We have not yet established the global structure of the spacetime, or the range of θ and ϕ. For example, $\mathcal{S}(\mathcal{P})$ looks locally like a 2-sphere, as we can see above, but nothing prevents it from alternatively having the topology of e.g. the space $\mathbb{R}P^2$ obtained by identifying antipodal points. "Spherical" stars or black holes of this type are not astrophysically relevant since they do not approach Minkowski-like geometry at large distances and hence such an object probably can't form. But this is not an issue one would find from purely local considerations such as those here.]

D. What about more symmetrical spacetimes?

The above considerations are not appropriate if $\mathcal{S}(\mathcal{P})$ has dimension 3, in which case there is no linear dependence such as Eq. (5). One might wonder if Eq. (12) applies in these cases. In fact it does – the basic reason is that such spacetimes are actually *more* symmetrical than spherical symmetry implies. One can use this to construct even more Killing fields, and then find a subspace of Killing fields satisfying Eq. (5). You will do this on the homework.

E. Reduction of the form of the metric

Equation (12) is in fact an overly general form for the metric, since we have treated x^0 and x^1 as arbitrary labels of the spheres consisting of "equivalent" points. We now impose a gauge transformation to simplify the form. To begin with, if r is not constant (the generic case), we may use it as one of our coordinates and denote the other coordinate by t. Then we have

$$ds^2 = g_{tt}dt^2 + 2g_{rt}dr\,dt + g_{rr}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{13}$$

Even this is too general – it is possible to use the remaining degree of freedom in defining "t" to eliminate g_{rt} – but we won't do this yet.

III. SPHERICALLY SYMMETRIC, TIME-STATIONARY SPACETIMES

We will now move on to spacetimes that are also time-stationary, i.e. possess a timelike Killing field $\boldsymbol{\xi}$ that commutes with the $\boldsymbol{J}_i$, i.e. $\mathcal{L}_{\boldsymbol{J}_i}\boldsymbol{\xi} = 0$. Considerations similar to those used above for the metric enable us to show that the components ξ^α are nonzero only for $\alpha = t, r$ and that they are independent of θ and ϕ. Moreover, even the ξ^r component must vanish, since $g_{\theta\theta} = r^2$ implies that

$$0 = \mathcal{L}_{\boldsymbol{\xi}} g_{\theta\theta} = 2r\xi^r. \tag{14}$$

So we have $\xi^\alpha \neq 0$ only for $\alpha = t$. A rescaling of t (i.e. choosing a new $t' = \int dt/\xi^t$ and then dropping the primes) sets $\xi^t = 1$. Then all components in Eq. (13) depend only on r.

Our final step in building the time-stationary metric is to eliminate g_{rt}. This can be done without upsetting $\xi^\alpha = (1,0,0,0)$ by defining

$$t' = t + f(r) \quad \leftrightarrow t = t' - f(r). \tag{15}$$

Then in the primed system we have

$$g_{rt'} = \frac{\partial x^\alpha}{\partial r}\frac{\partial x^\beta}{\partial t'} g_{\alpha\beta} = g_{rt} - f(r) g_{tt}. \tag{16}$$

So in the generic case, where $g_{tt} \neq 0$, it is possible to choose $f(r)$ so as to eliminate g_{rt}. Then the metric takes the remarkably simple form

$$ds^2 = g_{tt} dt^2 + g_{rr} dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2). \tag{17}$$

This is more often written as

$$ds^2 = -e^{2\Phi} dt^2 + e^{2\Lambda} dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2), \tag{18}$$

where $\Phi(r)$ and $\Lambda(r)$ are arbitrary functions of r. Equation (18) is what we will use in our study of spherical stars.

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[1] They are not of simply academic interest: the closed universe is an example.