Lecture XVI: Symmetrical spacetimes

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I. OVERVIEW

Our principal concern this term will be symmetrical solutions of Einstein's equations that are of astrophysical relevance. These are *relativistic stars*, white dwarfs and neutron stars, which we will approximate as spherical; *black holes*, the ultimate endpoint of evolution for some massive stars in addition to being found in the centers of galaxies; and *the Universe* itself.

Before we study these solutions, it is worth a digression to examine the role of symmetry in GR. We will consider here the case of spacetimes with continuous symmetries. (We won't investigate discrete symmetries in any formal way in this class, although we will use some of their intuitive properties later.) We will need to do this both to define a spherically symmetric object, and later on to study its perturbations.

The (optional) reading for this lecture is:

- MTW §25.2.

II. SOME PRELIMINARIES

Suppose we want to consider a system that is "time-independent" (or "time translation invariant" or *stationary*). In Newtonian physics or even in SR, it is clear what this means: one can choose a reference frame in which all quantities depend only on x^i and not on t, e.g. the density is $\rho(x^i,t) = \rho(x^i)$. But in GR the coordinates are free for us to choose, so we cannot simply transfer over this definition. In linearized GR, we circumvented the problem by making reference to the Minkowski background around which we perturb; but in full GR we must begin with a notion of "symmetry" that is not tied to a particular coordinate system.

A. Infinitesimal coordinate transformations and Lie derivatives

In general, a symmetry of any object is a mapping of the object to itself that preserves certain relevant properties (in our case, the metric). So let's consider a mapping $\mathcal{W}(\epsilon) : \mathcal{M} \rightarrow \mathcal{M}$, where ϵ parameterizes the continuous symmetry. We will force this mapping to be differentiable (with respect to both ϵ and $\mathcal{P}$) as many times as needed, and we will assume that at zero parameter the mapping is simply the identity:

$$[\mathcal{W}(0)](\mathcal{P}) = \mathcal{P}. \tag{1}$$

The prototype of this is the time translation invariance of Minkowski spacetime, where translating by a time ϵ takes a point (t, x^i) to

$$[\mathcal{W}(\epsilon)](t, x^i) = (t + \epsilon, x^i). \tag{2}$$

A continuous symmetry is often described by its behavior for infinitesimal values of the parameter; thus we define the vector field

$$\boldsymbol{\xi}(\mathcal{P}) = \frac{d}{d\epsilon}[\mathcal{W}(0)](\mathcal{P})\bigg|_{\epsilon=0}. \tag{3}$$

In our example of Minkowski space, the contravariant components are simply $\xi^\mu = (1, 0, 0, 0)$.

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Any such mapping can be thought of as a change of coordinates: given the mapping $\mathcal{W}(\epsilon)$, and an unprimed coordinate system, we can define the primed coordinates of any point $\mathcal{P}$ as the unprimed coordinates of $[\mathcal{W}(\epsilon)](\mathcal{P})$. The primed coordinate system depends on ϵ; in our example above, we have

$$x^{i'} = x^i \quad \text{and} \quad t' = t + \epsilon. \tag{4}$$

Now let's consider a tensor field $S^{\alpha_1...\alpha_M}{}_{\beta_1...\beta_N}$. In the primed system, it now has components:

$$S^{\alpha_1'...\alpha_M'}{}_{\beta_1'...\beta_N'}(x^{\mu'}) = \frac{\partial x^{\alpha_1'}}{\partial x^{\alpha_1}}...\frac{\partial x^{\alpha_M'}}{\partial x^{\alpha_M}}\frac{\partial x^{\beta_1}}{\partial x^{\beta_1'}}...\frac{\partial x^{\beta_N}}{\partial x^{\beta_N'}}S^{\alpha_1...\alpha_M}{}_{\beta_1...\beta_N}(x^\nu). \tag{5}$$

Note that the numerical value of any component here may in general depend on ϵ since the coordinate transformation depends on ϵ. But if $\mathcal{W}(\epsilon)$ really describes a symmetry and $\mathbf{S}$ is invariant under that transformation, there shouldn't be any such dependence. To make use of this fact for infinitesimal symmetries, we take the derivative of Eq. (5) with respect to ϵ at $\epsilon = 0$:

$$\begin{aligned} -\frac{\partial}{\partial\epsilon}S^{\alpha_1'...\alpha_M'}{}_{\beta_1'...\beta_N'}(x^{\mu'})\Big|_{\epsilon=0} &= -\xi^{\alpha_1'}{}_{,\gamma}S^{\gamma\alpha_2'...\alpha_M'}{}_{\beta_1'...\beta_N'}... - \xi^{\alpha_M'}{}_{,\gamma}S^{\alpha_1'...\alpha_{M-1}'\gamma}{}_{\beta_1'...\beta_N'} \\ &\quad +\xi^\gamma{}_{,\beta_1}S^{\alpha_1'...\alpha_M'}{}_{\gamma\beta_2'...\beta_N'}... + \xi^\gamma{}_{,\beta_N}S^{\alpha_1'...\alpha_M'}{}_{\beta_1'...\beta_{N-1}'\gamma} + \xi^\gamma S^{\alpha_1'...\alpha_M'}{}_{\beta_1'...\beta_N',\gamma}. \end{aligned} \tag{6}$$

Here we have differentiated each term in Eq. (5), and noted that at $\epsilon = 0$ all the Jacobians are Kronecker deltas: $\partial x^{\alpha_1'}/\partial x^{\alpha_1} = \delta^{\alpha_1'}{}_{\alpha_1}$, etc. The derivatives are simply

$$\frac{\partial}{\partial\epsilon}\frac{\partial x^{\alpha_1'}}{\partial x^{\alpha_1}}\Big|_{\epsilon=0} = \xi^{\alpha_1'}{}_{,\alpha_1} \quad \text{and} \quad \frac{\partial}{\partial\epsilon}\frac{\partial x^{\alpha_1}}{\partial x^{\alpha_1'}}\Big|_{\epsilon=0} = -\xi^{\alpha_1}{}_{,\alpha_1'}. \tag{7}$$

We have replaced the dummy indices in Eq. (6) with γs. Note that there is no confusion here of comparing tensors in the primed or unprimed system since at $\epsilon = 0$ they coincide.

The variation in Eq. (6) is so important that it gets a name: we call it the *Lie derivative*:

$$\begin{aligned} \mathcal{L}_{\boldsymbol{\xi}}S^{\alpha_1...\alpha_M}{}_{\beta_1...\beta_N} &= -\xi^{\alpha_1}{}_{,\gamma}S^{\gamma\alpha_2...\alpha_M}{}_{\beta_1...\beta_N}... - \xi^{\alpha_M}{}_{,\gamma}S^{\alpha_1...\alpha_{M-1}\gamma}{}_{\beta_1...\beta_N} \\ &\quad +\xi^\gamma{}_{,\beta_1}S^{\alpha_1...\alpha_M}{}_{\gamma\beta_2...\beta_N}... + \xi^\gamma{}_{,\beta_N}S^{\alpha_1...\alpha_M}{}_{\beta_1...\beta_{N-1}\gamma} + \xi^\gamma S^{\alpha_1...\alpha_M}{}_{\beta_1...\beta_N,\gamma}. \end{aligned} \tag{8}$$

It describes how the tensor $\mathbf{S}$ changes under an infinitesimal coordinate transformation $\boldsymbol{\xi}$. The rules for computing it are straightforward – one takes the partial derivative and then appends a correction term for each index – but note that it contains the derivative of $\boldsymbol{\xi}$.

The geometry of spacetime is invariant under a continuous symmetry with derivative $\boldsymbol{\xi}$ if and only if the Lie derivative of the metric is zero,

$$\mathcal{L}_{\boldsymbol{\xi}}\mathbf{g} = 0. \tag{9}$$

In this case, we say that $\boldsymbol{\xi}$ is a *Killing field.*

B. Properties of the Lie derivative

Note that we have made no use of the covariant derivative or metric in defining the Lie derivative. Indeed, the Lie derivative makes perfect sense on a manifold with no metric structure. But let us suppose that we have a metric. Since $\mathcal{L}_{\boldsymbol{\xi}}\mathbf{S}$ is a tensor, we may evaluate Eq. (8) in a local Lorentz frame, where all Christoffel symbols vanish, and conclude that

$$\begin{aligned} \mathcal{L}_{\boldsymbol{\xi}}S^{\alpha_1...\alpha_M}{}_{\beta_1...\beta_N} &= -\xi^{\alpha_1}{}_{;\gamma}S^{\gamma\alpha_2...\alpha_M}{}_{\beta_1...\beta_N}... - \xi^{\alpha_M}{}_{;\gamma}S^{\alpha_1...\alpha_{M-1}\gamma}{}_{\beta_1...\beta_N} \\ &\quad +\xi^\gamma{}_{;\beta_1}S^{\alpha_1...\alpha_M}{}_{\gamma\beta_2...\beta_N}... + \xi^\gamma{}_{;\beta_N}S^{\alpha_1...\alpha_M}{}_{\beta_1...\beta_{N-1}\gamma} + \xi^\gamma S^{\alpha_1...\alpha_M}{}_{\beta_1...\beta_N;\gamma}. \end{aligned} \tag{10}$$

[You can also prove by brute force expansions of Christoffel symbols that this reduces to Eq. (8).]

A consequence is that the Lie derivative of a scalar is simply the directional derivative,

$$\mathcal{L}_{\boldsymbol{\xi}}f = \nabla_{\boldsymbol{\xi}}f. \tag{11}$$

The Lie derivative of a vector is often called the *commutator*:

$$\mathcal{L}_{\boldsymbol{u}} v^\alpha = u^\beta v^\alpha{}_{,\beta} - v^\beta u^\alpha{}_{,\beta} = [\boldsymbol{u}, \boldsymbol{v}]^\alpha. \tag{12}$$

The name arises because of the following easily verified rule:

$$\mathcal{L}_{\boldsymbol{u}} \mathcal{L}_{\boldsymbol{v}} S^{\alpha_1 \ldots \alpha_M}{}_{\beta_1 \ldots \beta_N} - \mathcal{L}_{\boldsymbol{v}} \mathcal{L}_{\boldsymbol{u}} S^{\alpha_1 \ldots \alpha_M}{}_{\beta_1 \ldots \beta_N} = \mathcal{L}_{[\boldsymbol{u},\boldsymbol{v}]} S^{\alpha_1 \ldots \alpha_M}{}_{\beta_1 \ldots \beta_N}. \tag{13}$$

Clearly $[\boldsymbol{u}, \boldsymbol{v}] = -[\boldsymbol{v}, \boldsymbol{u}]$.

Finally one can show that the product rule holds:

$$\mathcal{L}_{\boldsymbol{\xi}} (A^\alpha{}_\beta B^\gamma{}_\delta) = (\mathcal{L}_{\boldsymbol{\xi}} A^\alpha{}_\beta) B^\gamma{}_\delta + A^\alpha{}_\beta \mathcal{L}_{\boldsymbol{\xi}} B^\gamma{}_\delta, \tag{14}$$

and that the Lie derivative commutes with contraction.

WARNING: The Lie derivative does *not* in general commute with the raising and lowering of indices, e.g. $\mathcal{L}_{\boldsymbol{\xi}} A^\mu$ is not the same vector as one obtains by raising the index of $\mathcal{L}_{\boldsymbol{\xi}} A_\mu$. Therefore it is necessary to specify whether the indices are up or down. The exception is when $\boldsymbol{\xi}$ is a Killing field.

III. SYMMETRIES

We may now describe a spacetime as having a continuous symmetry if Eq. (9) holds. This is most often described by expanding Eq. (10) and noting that $g_{\mu\nu;\gamma} = 0$; then we have

$$\mathcal{L}_{\boldsymbol{\xi}} g_{\mu\nu} = \xi^\gamma{}_{;\mu} g_{\gamma\nu} + \xi^\gamma{}_{;\nu} g_{\mu\gamma} = \xi_{\nu;\mu} + \xi_{\mu;\nu}. \tag{15}$$

Therefore $\boldsymbol{\xi}$ is a Killing field if and only if

$$\xi_{(\mu;\nu)} = 0. \tag{16}$$

It is easy to see that the Killing fields form a vector space – if $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are Killing fields, then $a\boldsymbol{\xi} + b\boldsymbol{\eta}$ is also a Killing field. This is no surprise: if e.g. t-translation is a symmetry, and x^1-translation is a symmetry, then a “diagonal” translation in both space and time must also be a symmetry. We may therefore obtain a description of all of the symmetries of a spacetime by finding a basis for the Killing fields and exploring the properties of each of the basis Killing fields. This is useful because all of the spacetimes we examine in practice have a finite number of linearly independent Killing fields.

A. Example: Minkowski spacetime

As a first example, let us classify all of the continuous symmetries of Minkowski spacetime. We work in standard Minkowski coordinates x^μ in which the Christoffel symbols vanish. Then the requirement of a Killing field is

$$\xi_{\mu,\nu} + \xi_{\nu,\mu} = 0. \tag{17}$$

Therefore the tensor $f_{\mu\nu} = \xi_{\mu,\nu}$ is antisymmetric. Moreover, we have

$$f_{\mu\nu,\sigma} = \xi_{\mu,\nu\sigma} = \xi_{\mu,\sigma\nu} = f_{\mu\sigma,\nu} = -f_{\sigma\mu,\nu}. \tag{18}$$

Thus $f_{\mu\nu,\sigma}$ flips sign if we cyclically permute the 3 indices. Repeating the above argument 3 times gives

$$f_{\mu\nu,\sigma} = -f_{\sigma\mu,\nu} = f_{\nu\sigma,\mu} = -f_{\mu\nu,\sigma}, \tag{19}$$

so we conclude that $f_{\mu\nu,\sigma} = 0$ and $f_{\mu\nu}$ is a constant. We can then integrate to find

$$\xi_\mu = f_{\mu\nu} x^\nu + b_\mu, \tag{20}$$

where $\boldsymbol{b}$ is any 4-vector. Their components are all affine functions of x^μ.

Equation (20) then implies that the Killing fields form a 10-dimensional vector space (4 components of b_μ and 6 of $f_{\mu\nu}$). These can be written in the following basis:

- The 4 *translations*, which are simply the basis vectors: $\boldsymbol{\pi}_\alpha = \boldsymbol{e}_\alpha$.

- The 3 *rotations*, $(J_1)^\mu = (0, 0, -x^3, x^2)$, $(J_2)^\mu = (0, x^3, 0, -x^1)$, and $(J_3)^\mu = (0, -x^2, x^1, 0)$. [*Note* that the subscripts on J_i indicate which Killing vector, not which component.]

- The 3 *boosts*, $(K_1)^\mu = (x^1, x^0, 0, 0)$, etc.

These are in fact the familiar transformations of SR between inertial frames. Most of the symmetries we will study in GR are appropriate generalizations of these. Their designation as a particular type of symmetry will depend on other mathematical properties (specifically their commutators).

You know from introductory physics that rotations do not commute: an infinitesimal rotation around the 1-axis followed by an infinitesimal rotation around the 2-axis is not the same as rotating around 2 and then 1. Mathematically, this is incorporated in the commutator,

$$[\boldsymbol{J}_1, \boldsymbol{J}_2] = -\boldsymbol{J}_3. \tag{21}$$

I'll prove just the 1-component of this equation,

$$[\boldsymbol{J}_1, \boldsymbol{J}_2]^1 = (J_1)^\beta (J_2)^1{}_{,\beta} - (J_2)^\beta (J_1)^1{}_{,\beta} = (J_1)^3 (J_2)^1_{,3} = x^2 (x^3)_{,3} = x^2. \tag{22}$$

Generally we have

$$[\boldsymbol{J}_i, \boldsymbol{J}_j] = -\epsilon_{ijk} \boldsymbol{J}_k. \tag{23}$$

This relation is probably familiar from quantum mechanics, aside from the factor of i that distinguishes the infinitesimal rotation from the quantum angular momentum operator. It is possible to establish *commutation relations* of this sort for all of the Minkowski Killing fields; for example

$$[\boldsymbol{J}_i, \boldsymbol{\pi}_0] = 0 \tag{24}$$

and

$$[\boldsymbol{J}_i, \boldsymbol{\pi}_j] = -\epsilon_{ijk} \boldsymbol{\pi}_k. \tag{25}$$

B. Commutation relations

The existence of commutation relations is generic to continuous symmetries (in GR and otherwise). Given any two Killing vectors $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$, it is easily seen that their commutator must be a Killing field, since

$$\mathcal{L}_{[\boldsymbol{\xi}, \boldsymbol{\eta}]} \mathbf{g} = \mathcal{L}_{\boldsymbol{\xi}} \mathcal{L}_{\boldsymbol{\eta}} \mathbf{g} - \mathcal{L}_{\boldsymbol{\eta}} \mathcal{L}_{\boldsymbol{\xi}} \mathbf{g} = 0 - 0 = 0. \tag{26}$$

Furthermore, the commutator satisfies a Jacobi identity,

$$[\boldsymbol{\xi}, [\boldsymbol{\eta}, \boldsymbol{\zeta}]] + [\boldsymbol{\eta}, [\boldsymbol{\zeta}, \boldsymbol{\xi}]] + [\boldsymbol{\zeta}, [\boldsymbol{\xi}, \boldsymbol{\eta}]] = 0 \tag{27}$$

(each combination such as $\mathcal{L}_{\boldsymbol{\xi}} \mathcal{L}_{\boldsymbol{\eta}} \mathcal{L}_{\boldsymbol{\zeta}}$ appears twice with opposite signs). A vector space that satisfies Eq. (27) is said to be a *Lie algebra*. Clearly a Lie algebra is defined by the commutators of the basis fields. The familiar angular momentum operators of quantum mechanics satisfy a Lie algebra, given by Eq. (23) (again with the is located in different places).

We are now ready to make a definition:

- A spacetime is *stationary* if it possesses a Killing field $\boldsymbol{\pi}_0$ that is timelike in some portion of spacetime. (Normally we will use this term to describe nonrotating stars and black holes; in the latter case, $\boldsymbol{\pi}_0$ becomes null on the horizon and is spacelike inside the hole, so we do not want to impose a restriction of $\boldsymbol{\pi}_0$ being timelike everywhere.)

- A spacetime is *spherically symmetric* if it possesses 3 Killing fields $\boldsymbol{J}_1$, $\boldsymbol{J}_2$, and $\boldsymbol{J}_3$ satisfying Eq. (23).

- A spacetime is *homogeneous and isotropic* if it possesses 3 "rotation" Killing fields $\boldsymbol{J}_i$ and 3 "translation" Killing fields $\boldsymbol{\pi}_i$ satisfying Eq. (23) and (25).

Normally situations with more Killing fields are easier to analyze since they are more symmetrical. Thus our major efforts will be devoted to spherical stars (4 Killing fields) and cosmology (6 Killing fields). The most difficult solution this term is the Kerr black hole (2 Killing fields).

C. Conservation laws

In ordinary mechanics, we know that a continuous symmetry is associated with a corresponding conservation law. The same is true in GR. If we take a Killing field $\boldsymbol{\xi}$, and then trace a freely falling particle of momentum $\boldsymbol{p}$ along its trajectory, we find

$$\frac{d}{d\lambda}(\boldsymbol{p}\cdot\boldsymbol{\xi}) = \frac{D\boldsymbol{p}}{d\lambda}\cdot\boldsymbol{\xi} + \boldsymbol{p}\cdot\nabla_{\boldsymbol{p}}\boldsymbol{\xi} = 0 + p^\alpha p^\beta \xi_{\beta;\alpha} = 0. \tag{28}$$

Therefore $\boldsymbol{p}\cdot\boldsymbol{\xi}$ is conserved along a trajectory.

The examples from Minkowski spacetime are that

- The $\boldsymbol{\pi}_\alpha$ Killing fields tell us that $\boldsymbol{p}\cdot\boldsymbol{\pi}_\alpha = \boldsymbol{p}\cdot\boldsymbol{e}_\alpha = p_\alpha$ is conserved – thus we have conservation of *energy* and *3-momentum*.
- The J_i Killing fields tell us that e.g. $-L_1 \equiv \boldsymbol{p}\cdot\boldsymbol{J}_1 = -p_2x^3 + p_3x^2$ is conserved – thus we have *conservation of angular momentum*.
- The K_i Killing fields tell us that $Y_i \equiv -\boldsymbol{p}\cdot\boldsymbol{K}_i = -p_0x^i + -p_ix^0 = Ex^i - p^it$ is conserved – thus the particle's position moves at a rate $dx^i/dt = p^i/E$.

In less symmetrical spacetimes, subsets of these conservation laws are valid. For example, in stationary and spherically symmetric spacetimes (such as the exterior of a nonrotating spherical star) we have a conserved energy $E = -\boldsymbol{p}\cdot\boldsymbol{\xi}$ (where $\boldsymbol{\xi}$ is the time-translation Killing field) and 3 conserved angular momenta L_i.

D. Electrodynamics

In the case where there is also an electromagnetic field obeying the symmetry, it is possible to extend the conservation laws to a charged particle that is not following a geodesic. Specifically, we consider a field tensor $F_{\mu\nu}$. Since this field satisfies the Maxwell equation $F_{[\mu\nu,\sigma]} = 0$, it is possible at least locally to write it as the exterior derivative of a potential,

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} = A_{\nu;\mu} - A_{\mu;\nu}. \tag{29}$$

Note that $\boldsymbol{A}$ is not unique, as we may always transform $A_\mu \to A_\mu + \partial_\mu\Lambda$ for any scalar Λ with no effect on $F_{\mu\nu}$.

If a particle has charge e and accelerates only under the influence of the electromagnetic force, so that $Dp^\alpha/d\lambda = eF^{\alpha\beta}p_\beta$, then

$$\frac{d}{d\lambda}(\boldsymbol{p}\cdot\boldsymbol{\xi}) = eF^{\alpha\beta}p_\beta\xi_\alpha + \boldsymbol{p}\cdot\nabla_{\boldsymbol{p}}\boldsymbol{\xi} = eF^{\alpha\beta}p_\beta\xi_\alpha. \tag{30}$$

However

$$\frac{d}{d\lambda}(\boldsymbol{A}\cdot\boldsymbol{\xi}) = A^\alpha p^\beta\xi_{\alpha;\beta} + \xi^\alpha p^\beta A_{\alpha;\beta} = A_\alpha p^\beta \xi^\alpha{}_{;\beta} + \xi^\alpha p^\beta A_{\beta;\alpha} - \xi^\alpha p^\beta F_{\alpha\beta} = p^\beta \mathcal{L}_{\boldsymbol{\xi}} A_\beta - \xi^\alpha p^\beta F_{\alpha\beta}. \tag{31}$$

Therefore if the electromagnetic potential is also symmetrical under the transformation generated by $\boldsymbol{\xi}$ – i.e. if $\mathcal{L}_{\boldsymbol{\xi}}A_\beta = 0$ – then the first term vanishes and we have

$$\frac{d}{d\lambda}[(\boldsymbol{p} + e\boldsymbol{A})\cdot\boldsymbol{\xi}] = 0 \tag{32}$$

and so

$$(\boldsymbol{p} + e\boldsymbol{A})\cdot\boldsymbol{\xi} = \text{constant}. \tag{33}$$