Lecture XV: Gravitational energy and orbital decay by gravitational radiation

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(Dated: December 2, 2011)

I. OVERVIEW

The previous discussions have taken place in the context of linearized GR, which is not a fully consistent theory. We will now discuss some aspects of GR in the nonlinear regime, with particular attention to isolated systems. Our eventual goal is to compute the energy loss of a system emitting gravitational waves. We introduced the concepts in the last lecture, and are now ready to proceed to the calculation.

The recommended reading for this lecture is:

- MTW §20.4,20.5.

II. GRAVITATIONAL ENERGY OF A SYSTEM

Our first task is to determine the gravitational energy of a system. Put simply: given a collection of objects $\{A\}$ of mass m_A – for example, stars in a galaxy, or more fundamentally electrons and ions in a star – how does one determine the total mass? In Newtonian gravity, the answer is

$$M = \sum_A m_A, \tag{1}$$

but we will see in general relativity that it is not.

In order to make life simpler, we will make the following assumptions:

- The self-gravity of individual consituents can be neglected. This is a good approximation for e.g. a proton in the Sun, whose potential well depth $m_p/r_p = Gm_p/r_pc^2 \sim 10^{-43}$, versus the Sun's potential well depth of $GM_\odot/R_\odot c^2 = 2\times 10^{-6}$. (Of course, if the constituents are really pointlike particles, this description doesn't work. In full GR, the only consistent description of a "point" particle is a black hole. In the case of an elementary particle of mass $< m_{\rm Planck} \sim 10^{-5}$ g, there is no meaningful description of its gravitational field at distances smaller than the Compton wavelength.)
- The objects are slowly moving and gravity is weak. In particular, we work to order v^2 in the velocities and Φ in the potential. This makes sense because for virialized systems, v^2 and Φ are typically of the same order.

The total mass of the system is simply its energy measured in the center of mass frame. That is, we are interested in the energy

$$E = \int T^{{\rm eff}\,00}\,d^3x \tag{2}$$

and momentum $P^i = \int T^{{\rm eff}\,0i}\,d^3x$. Then the mass is

$$M = \sqrt{E^2 - (P^i)^2}. \tag{3}$$

We note that in the "naive" center-of-mass frame, obtained by setting

$$\sum_A m_A \boldsymbol{v}_A = 0, \tag{4}$$

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the momentum becomes at least of order $\sim Mv^2$, so $(P^i)^2$ is of order $\sim M^2v^4$. Therefore it is of high order and we may take

$$M = E = \int T^{\mathrm{eff}\,00}\,d^3x = \int T^{00}\,d^3x + \int t^{00}\,d^3x. \tag{5}$$

This integral manifestly has two parts: an integral of the 00 component of the stress-energy tensor over the coordinate volume, and an integral over the effective gravitational energy.

A. The first integral

To obtain the first integral in Eq. (5), we need the formula for the energy density of a particle. In special relativity, this was

$$T^{00}(t, x^i) = p^0\delta^{(3)}[x^i - y^i(t)] = \frac{m_A}{\sqrt{1-\boldsymbol{v}_A^2}}\delta^{(3)}[x^i - y^i_A(t)]. \tag{6}$$

In GR this gets modified. The above equation should be true in a local Lorentz frame – *any* local Lorentz frame – but the coordinate frame is not of this type. Instead, we recall the metric for a system of slow-moving particles to first order,

$$ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2]. \tag{7}$$

Then at the instantaneous position of a particle, we may define the local Lorentz frame of an observer at rest relative to the coordinate system,

$$\hat{t} = (1+\Phi)(t - t_{\mathrm{orig}}), \quad \hat{x}^i = (1-\Phi)(x^i - x^i_{\mathrm{orig}}), \tag{8}$$

where one can readily see that at the origin $(t_{\mathrm{orig}}, x^i_{\mathrm{orig}})$ the metric is $-d\hat{t}^2 + (d\hat{x}^i)^2$. If we choose the spatial origin of the local Lorentz frame to be at the position of the particle at some time t_{orig}, then we have

$$T^{\hat{0}\hat{0}} = \frac{m_A}{\sqrt{1-\boldsymbol{v}_A^2}}\delta^{(3)}(\hat{x}^i). \tag{9}$$

Converting to the original coordinate system involves two factors of $dt/d\hat{t} = 1 - \Phi$, and the Jacobian $|d^3x^i/d^3\hat{x}^j| = 1 + 3\Phi$. Furthermore, to relevant order we may Taylor-expand the inverse square root, yielding

$$T^{00}(t, x^i) = m_A\left[1 + \frac{1}{2}(\boldsymbol{v}_A)^2 + \Phi\right]\delta^{(3)}[x^i - y^i_A(t)]. \tag{10}$$

Integration is trivial due to the δ-function, and we get that the first integral is

$$\int T^{00}\,d^3x = \sum_A m_A\left[1 + \frac{1}{2}(\boldsymbol{v}_A)^2 + \Phi(\boldsymbol{y}_A)\right]. \tag{11}$$

B. The second integral

The second integral is an exercise in nonlinear general relativity. Fortunately, just as for gravitational waves, we can use the linear theory prediction for the metric to compute

$$t^{\mu\nu} = -\frac{1}{8\pi}G^{\mu\nu} + \frac{1}{16\pi}H^{\mu\alpha\nu\beta}{}_{,\alpha\beta} \tag{12}$$

since this is a second-order function of the metric perturbation.

Since our interest is in the t^{00} component, we need G^{00}. To simplify our calculations further, we begin by considering which terms we will need to keep. If the system has total mass $\sim M$ and size $\sim L$, with $\Phi \sim v^2 \sim M/L$, then we want to keep terms in the total energy through order $\sim Mv^2 \sim L\Phi^2$. This means we need terms in the effective energy density through order $\sim \Phi^2/L^2$. Now the timescale for evolution of the system is $T \sim L/v$. All terms in G^{00} will have two derivatives of Φ or Φ^2 by dimensional analysis, so of the linear and quadratic terms, we need to:

- Keep terms of order $\nabla^2\Phi \sim \Phi/L^2$.
- Keep terms of order $\dot{\Phi}\nabla\Phi \sim \Phi/LT$ – **except** that there can't be any by rotational symmetry.
- Keep terms of order $\ddot{\Phi} \sim \Phi/T^2 \sim \Phi v^2/L^2 \sim \Phi^2/L^2$.
- Keep terms of order $\Phi\nabla^2\Phi \sim (\nabla\Phi)^2 \sim \Phi^2/L^2$.
- Drop terms of order $\dot{\Phi}\nabla\Phi$, $\Phi\ddot{\Phi}$, $\dot{\Phi}^2$, etc. as these are of order $\Phi^2 v/L^2$ or $\Phi^2 v^2/L^2$.

Thus the only terms in G^{00} that we need to keep that depend on time derivatives of Φ are the first-order $\ddot{\Phi}$ terms. It then follows that as far as the computation of second-order terms is concerned, we can drop the time dependence of Φ entirely! So we will do this and proceed.

We may easily calculate the H-term: we have

$$\bar{h}^{00} = -4\Phi, \quad \text{all other entries zero.} \tag{13}$$

Then we find that

$$H^{0i0j} = -\bar{h}^{00}\eta^{ij} - \eta^{00}\bar{h}^{ij} + \bar{h}^{i0}\eta^{0j} + \eta^{i0}\bar{h}^{0j} = 4\Phi\delta^{ij}, \tag{14}$$

and so

$$H^{0\alpha0\beta}{}_{,\alpha\beta} = H^{0i0j}{}_{,ij} = 4\Phi_{,ij}\delta^{ij} = 4\nabla^2\Phi. \tag{15}$$

Therefore we have simply

$$t^{00} = -\frac{1}{8\pi}G^{00} + \frac{1}{4\pi}\nabla^2\Phi. \tag{16}$$

We obtain the Christoffel symbols to second order in Φ,

$$\begin{aligned}
\Gamma^0{}_{00} &= 0, \\
\Gamma^0{}_{0i} &= (1-2\Phi)\Phi_{,i}, \\
\Gamma^0{}_{ij} &= 0, \\
\Gamma^i{}_{00} &= (1+2\Phi)\Phi_{,i}, \\
\Gamma^i{}_{0j} &= 0, \quad \text{and} \\
\Gamma^i{}_{jk} &= -(1+2\Phi)(\Phi_{,k}\delta_{ij} + \Phi_{,j}\delta_{ik} - \Phi_{,i}\delta_{jk}).
\end{aligned} \tag{17}$$

Then we can obtain the Ricci tensor components,

$$R_{\mu\nu} = \Gamma^\alpha{}_{\mu\nu,\alpha} - \Gamma^\alpha{}_{\mu\alpha,\nu} + \Gamma^\alpha{}_{\beta\alpha}\Gamma^\beta{}_{\mu\nu} - \Gamma^\alpha{}_{\beta\nu}\Gamma^\beta{}_{\alpha\mu}. \tag{18}$$

Again, it is convenient to find the combinations

$$\Gamma^\alpha{}_{0\alpha} = 0 \quad \text{and} \quad \Gamma^\alpha{}_{k\alpha} = -2(1+4\Phi)\Phi_{,k}. \tag{19}$$

Since we only need G^{00} and the metric is diagonal, we only need to find R_{00} and R_{mm}. These are:

$$\begin{aligned}
R_{00} &= \partial_i[(1+2\Phi)\Phi_{,i}] - [0] + [-2\Phi_{,k}\Phi_{,k}] - [2\Phi_{,i}\Phi_{,i}] \\
&= \nabla^2\Phi + 2\Phi\nabla^2\Phi - 2\Phi_{,i}\Phi_{,i}
\end{aligned} \tag{20}$$

and

$$\begin{aligned}
R_{mm} &= \partial_i[(1+2\Phi)\Phi_{,i}] - \partial_m[-2(1+4\Phi)\Phi_{,m}] + [-2\Phi_{,k}\Phi_{,k}] - [0] \\
&= 3\nabla^2\Phi + 8\Phi_{,i}\Phi_{,i} + 10\Phi\nabla^2\Phi.
\end{aligned} \tag{21}$$

The trace is

$$R = -(1-2\Phi)R_{00} + (1+2\Phi)R_{mm} = 2\nabla^2\Phi + 10\Phi_{,i}\Phi_{,i} + 16\Phi\nabla^2\Phi, \tag{22}$$

and hence

$$G_{00} = R_{00} + \frac{1}{2}(1 + 2\Phi)R = 2\nabla^2\Phi + 12\Phi\nabla^2\Phi + 3\Phi_{,i}\Phi_{,i}. \tag{23}$$

Then we find

$$G^{00} = (1 - 4\Phi)G_{00} = 2\nabla^2\Phi + 4\Phi\nabla^2\Phi + 3\Phi_{,i}\Phi_{,i}. \tag{24}$$

It follows that the only nonlinear term is the gradient of Φ:

$$t^{00} = -\frac{1}{2\pi}\Phi\nabla^2\Phi - \frac{3}{8\pi}(\nabla\Phi)^2. \tag{25}$$

Thus there is in some sense an "effective energy density" of the gravitational field, but again beware: like the energy density of gravitational waves, t^{00} is **not** a measurable energy density in any sense. It has a total energy,

$$\int t^{00}\, d^3x = \int \left[-\frac{1}{2\pi}\Phi\nabla^2\Phi - \frac{3}{8\pi}(\nabla\Phi)^2\right] d^3x. \tag{26}$$

This time, even this integral is not by itself meaningful, because there is matter present: one can only measure the total mass of the system, and cannot separate the "integral of T^{00}" from the "effective energy of gravity." Equation (26) only attains physical meaning when plugged into Eq. (5).

We may use integration by parts to simplify Eq. (26):

$$\int t^{00}\, d^3x = -\frac{1}{8\pi}\int \Phi\nabla^2\Phi\, d^3x. \tag{27}$$

Using the rule that $\nabla^2\Phi = 4\pi\rho$ to lowest order, and using that we have a collection of masses, we find

$$\int t^{00}\, d^3x = -\frac{1}{2}\sum_A m_A \Phi[y^i_A(t)]. \tag{28}$$

C. The total mass

From Eq. (5), we now find a total mass of

$$M = \sum_A m_A\left[1 + \frac{1}{2}(\boldsymbol{v}_A)^2 + \Phi(y^i_A)\right] - \frac{1}{2}\sum_A m_A\Phi(y^i_A). \tag{29}$$

The terms involving the potential can be combined,

$$M = \sum_A m_A\left[1 + \frac{1}{2}(\boldsymbol{v}_A)^2\right] + \frac{1}{2}\sum_A m_A\Phi(y^i_A). \tag{30}$$

Finally, using the linear Newtonian theory for the estimate of the potentials, we have

$$M = \sum_A m_A\left[1 + \frac{1}{2}(\boldsymbol{v}_A)^2\right] + \frac{1}{2}\sum_{A,B} m_A\frac{-m_B}{r_{AB}} \tag{31}$$

or (reducing the sum to avoid any double-counting)

$$M = \sum_A m_A\left[1 + \frac{1}{2}(\boldsymbol{v}_A)^2\right] - \sum_{A<B}\frac{m_A m_B}{r_{AB}}. \tag{32}$$

This is the total gravitational mass of the system, measurable via e.g. Kepler's third law, to order $M\Phi$ or Mv^2.

Equation (32) should be familiar: it is the total rest mass of the system, corrected by the contribution of kinetic energy and the Newtonian formula for gravitational energy.

But what does this equation mean? It is in stark contradiction to Newton's theory of gravity! In Newtonian gravity, a binary star consisting of two 1 $M_\odot$ stars has the same gravity (if observed from far enough away) as a 2 $M_\odot$ star. In Einstein's theory, this is no longer the case. The negative binding energy of the binary star, combining both the kinetic and potential terms, also gravitates. The resulting binary has a measurable gravitational mass of *less* than 2 $M_\odot$.

III. APPLICATION: INSPIRAL OF A BINARY STAR

As a final application, let us consider the evolution of a binary star composed of two components with masses M_1 and M_2 with separation a on a circular orbit. We will make the velocities involved nonrelativisitic. The system has a kinetic+potential energy of

$$E_{\rm orb} = -\frac{M_1 M_2}{2a} \tag{33}$$

and hence a total mass of

$$M = M_1 + M_2 - \frac{M_1 M_2}{2a}. \tag{34}$$

The orbital frequency of the system is

$$\Omega \equiv \frac{2\pi}{P} = \frac{(M_1+M_2)^{1/2}}{a^{3/2}}. \tag{35}$$

Our interest is in following the effect of gravitational radiation on the orbit. To do this, we first need to find the quadrupole moment. For masses separated at angle $\phi = \phi_0 + \Omega t$, this is

$$Q_{ij} = \frac{M_1 M_2}{M_1+M_2} a^2 \begin{pmatrix} \cos^2\phi - \frac{1}{3} & \cos\phi\sin\phi & 0 \\ \cos\phi\sin\phi & \sin^2\phi - \frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}. \tag{36}$$

If we use the double-angle identities, this becomes

$$Q_{ij} = \frac{M_1 M_2}{M_1+M_2} a^2 \begin{pmatrix} \frac{1}{2}\cos 2\phi + \frac{1}{6} & \frac{1}{2}\sin 2\phi & 0 \\ \frac{1}{2}\sin 2\phi & -\frac{1}{2}\cos 2\phi - \frac{1}{6} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}, \tag{37}$$

and taking the third derivative gives

$$\dddot{Q}_{ij} = \frac{M_1 M_2}{M_1+M_2} a^2 \Omega^3 \begin{pmatrix} 4\sin 2\phi & 4\cos 2\phi & 0 \\ 4\cos 2\phi & -4\sin 2\phi & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{38}$$

The gravitational wave power is then

$$-\langle \dot{E} \rangle = \frac{1}{5}\langle \dddot{Q}_{ij}\dddot{Q}_{ij}\rangle = \frac{1}{5}\left(\frac{M_1 M_2}{M_1+M_2} a^2 \Omega^3\right)^2 [32] = \frac{32}{5}\left(\frac{M_1 M_2}{M_1+M_2}\right)^2 a^4 \Omega^6. \tag{39}$$

Using Kepler's second law to eliminate Ω gives

$$-\langle \dot{E} \rangle = \frac{32}{5}\frac{M_1^2 M_2^2 (M_1+M_2)}{a^5}. \tag{40}$$

This is the rate at which the system loses orbital energy. Assuming that the masses of the objects don't change (e.g. that there is no transfer of energy from the internal structure of the bodies into the orbit), we may equate this with the rate of change of orbital energy,

$$\langle \dot{E} \rangle = \partial_t \left(M_1 + M_2 - \frac{M_1 M_2}{2a} \right) = \frac{M_1 M_2}{2a^2}\dot{a}, \tag{41}$$

and hence obtain

$$\dot{a} = -\frac{64}{5}\frac{M_1 M_2 (M_1+M_2)}{a^3}. \tag{42}$$

The $-$ sign indicates that the two bodies spiral together.

Since the rate of inspiral due to gravitational wave emission is proportional to a^{-3}, it follows that as the two bodies approach each other, they inspiral faster and faster. One may find the approach time by taking

$$\partial_t(a^4) = 4a^3\dot{a} = -\frac{256}{5} M_1 M_2 (M_1+M_2), \tag{43}$$

and hence we see that the inspiral reaches $a = 0$ in a finite time

$$t_{\rm GW} = \frac{5a^4}{256 M_1 M_2 (M_1+M_2)}. \tag{44}$$

This time is shortest for massive bodies on close orbits, as one might expect.

A. Examples

As a simple example, let's consider the inspiral times associated with solar-system scales. Recall that, converted into times, a solar mass is 4.9 μs and the astronomical unit is 500 s. Therefore, we can calculate the inspiral time of a system:

$$t_{\rm GW} = 3.3 \times 10^{17}\,{\rm yr}\,\frac{(a/1\,{\rm AU})^4}{M_1 M_2 (M_1 + M_2)/M_\odot^3}. \tag{45}$$

For the Earth orbiting the Sun, with $M_1 = M_\odot$ and $M_2 = 3 \times 10^{-6} M_\odot$ at a separation of 1 AU, the inspiral time is 10^{23} years. Of course by then the Sun will have turned into a white dwarf, Mercury and (maybe) Venus and Earth will have been consumed, and it is doubtful even that the orbits of the other planets are stable over that timescale. As a more extreme example one could consider the "hot Jupiters" that have been found around other stars with $M_1 \sim 10^{-3} M_\odot$ and $a = 0.05$ AU. There the inspiral time is 2×10^{15} years. So we can see that even in extreme situations, gravitational waves have no effect on planetary orbits.

Gravitational waves do however have a more significant effect on binary stars. If we consider a binary with masses of $M_1 = M_2 = M_\odot$, and we ask how close the orbits must be to merge in less than the age of the Universe (10^{10} years), we find

$$a < 0.016\,{\rm AU} \quad {\rm or} \quad P < 12\,{\rm hr}. \tag{46}$$

There are many instances of stellar remnants (white dwarfs and neutron stars) in orbits with periods of this order of magnitude or shorter (even as short as a few minutes). Such objects will spiral in due to gravitational wave emission and lead to mergers, which will be detectable as bursts of gravitational waves by the next generation of detectors.