Lecture XIV: The "energy" of gravitational waves

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I. OVERVIEW

The previous discussions have taken place in the context of linearized GR, which is not a fully consistent theory. We will now discuss some aspects of GR in the nonlinear regime, with particular attention to isolated systems. Our eventual goal is to compute the energy loss of a system emitting gravitational waves. We introduced the concepts in the last lecture, and are now ready to proceed to the calculation.

The recommended reading for this lecture is:

- MTW §20.4,20.5.

II. PRELIMINARIES

Recall that the rate of energy loss from a source emitting gravitational waves was given by

$$\dot{E} = -\int_{\partial\mathcal{V}} t^{0i} n_i \, d^2x, \tag{1}$$

where the integral is taken over a surface enclosing the source (here we will use a sphere of radius R), and the "effective" stress-energy tensor of the metric is

$$t^{\mu\nu} = -\frac{1}{8\pi} G^{\mu\nu} + \frac{1}{16\pi} H^{\mu\alpha\nu\beta}{}_{,\alpha\beta}. \tag{2}$$

The last piece cancels the **linear** contributions to $G^{\mu\nu}$ as a function of $h^{\mu\nu}$. The problem then is to compute Eq. (2) – or at least its time average – for a gravitational wave. We will work to second order in our calculation of $t^{\mu\nu}$; the first-order piece vanishes by construction, so this is the leading contribution.

III. CALCULATION

We will consider here a plane-parallel gravitational wave of the form

$$ds^2 = -dt^2 + [1 + 2h_+(t - x^3)](dx^1)^2 + 4h_\times(t - x^3)\, dx^1 dx^2 + [1 - 2h_+(t - x^3)](dx^2)^2 + (dx^3)^2 \tag{3}$$

propagating in the 3-direction. Since we are interested in $t^{\mu\nu}$, which by construction is 2nd order in the metric perturbations, we only need the input metric to be correct to first order – i.e. we don't need to worry that Eq. (3) is only a solution of Einstein's equations to first order.

Here the strains h_+ and $h_\times$ are functions of $t - x^3$ and contribute to the metric perturbation,

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2h_+ & 2h_\times & 0 \\ 0 & 2h_\times & -2h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4}$$

Since $h = 0$, there is no distinction now between $h_{\mu\nu}$ and $\bar{h}_{\mu\nu}$.

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A. Christoffel symbols

We first need the Christoffel symbols. The partial derivatives of the metric are (to all orders)

$$g_{11,0} = -g_{11,3} = -g_{22,0} = g_{22,3} = 2\dot{h}_+ \quad \text{and} \quad g_{12,0} = -g_{12,3} = 2\dot{h}_\times, \tag{5}$$

and the nonzero inverse metric components are

$$g^{33} = 1, \quad g^{00} = -1, \quad g^{11} = 1 - 2h_+ - 4(h_+^2 + h_\times^2), \quad g^{22} = 1 + 2h_+ - 4(h_+^2 + h_\times^2), \quad \text{and} \quad g^{12} = -2h_\times. \tag{6}$$

Therefore we find the following nonzero Christoffel symbols:

$$\begin{aligned}
\Gamma^0{}_{11} &= \dot{h}_+, \\
\Gamma^0{}_{12} &= \dot{h}_\times, \\
\Gamma^0{}_{22} &= -\dot{h}_+, \\
\Gamma^3{}_{11} &= \dot{h}_+, \\
\Gamma^3{}_{12} &= \dot{h}_\times, \\
\Gamma^3{}_{22} &= -\dot{h}_+, \\
\Gamma^1{}_{10} &= \dot{h}_+ - 2h_+\dot{h}_+ - 2h_\times\dot{h}_\times, \\
\Gamma^1{}_{20} &= \dot{h}_\times - 2h_+\dot{h}_\times + 2h_\times\dot{h}_+, \\
\Gamma^2{}_{10} &= \dot{h}_\times + 2h_+\dot{h}_\times - 2h_\times\dot{h}_+, \\
\Gamma^2{}_{20} &= -\dot{h}_+ - 2h_+\dot{h}_+ - 2h_\times\dot{h}_\times, \\
\Gamma^1{}_{13} &= -\dot{h}_+ + 2h_+\dot{h}_+ + 2h_\times\dot{h}_\times, \\
\Gamma^1{}_{23} &= -\dot{h}_\times + 2h_+\dot{h}_\times - 2h_\times\dot{h}_+, \\
\Gamma^2{}_{13} &= -\dot{h}_\times - 2h_+\dot{h}_\times + 2h_\times\dot{h}_+, \quad \text{and} \\
\Gamma^2{}_{23} &= \dot{h}_+ + 2h_+\dot{h}_+ + 2h_\times\dot{h}_\times.
\end{aligned} \tag{7}$$

B. Ricci and Einstein tensors

The Ricci tensor is obtained via

$$R_{\mu\nu} = \Gamma^\alpha{}_{\mu\nu,\alpha} - \Gamma^\alpha{}_{\mu\alpha,\nu} + \Gamma^\alpha{}_{\beta\alpha}\Gamma^\beta{}_{\mu\nu} - \Gamma^\alpha{}_{\beta\nu}\Gamma^\beta{}_{\alpha\mu}. \tag{8}$$

We recall again that partial derivatives with respect to x^1 and x^2 vanish, and those with respect to x^3 have $f_{,3} = -\dot{f}$ for any object f that is a function of only $t - x^3$.

We also note the auxiliary results

$$\Gamma^\alpha{}_{0\alpha} = -4h_+\dot{h}_+ - 4h_\times\dot{h}_\times, \quad \Gamma^\alpha{}_{3\alpha} = 4h_+\dot{h}_+ + 4h_\times\dot{h}_\times, \quad \text{and} \quad \Gamma^\alpha{}_{1\alpha} = \Gamma^\alpha{}_{2\alpha} = 0. \tag{9}$$

Then we find

$$R_{00} = -\partial_t(-4h_+\dot{h}_+ - 4h_\times\dot{h}_\times) - 2\dot{h}_+^2 - 2\dot{h}_\times^2 = 4h_+\ddot{h}_+ + 2\dot{h}_+^2 + 4h_\times\ddot{h}_\times + 2\dot{h}_\times^2. \tag{10}$$

Since a replacement of a lower 0 with a lower 3 in the Christoffel symbols always gives a $-$ sign, we have

$$R_{03} = -R_{00} \quad \text{and} \quad R_{33} = R_{00}. \tag{11}$$

The mixed Ricci tensor components are

$$R_{01} = R_{02} = R_{31} = R_{32} = 0 \tag{12}$$

by symmetry. It is less obvious, but it turns out that the other components also vanish, $R_{11} = R_{12} = R_{22} = 0$. For e.g. R_{11} this is because, of the terms in Eq. (8), we have: (i) $\Gamma^0{}_{11} = \Gamma^3{}_{11}$, but the partial derivatives with respect to t and x^3 give opposite signs; (ii) $\Gamma^\alpha{}_{1\alpha} = 0$; (iii) $\Gamma^\beta{}_{11} = 0$; and (iv) in $\Gamma^\alpha{}_{\beta 1}\Gamma^\beta{}_{\alpha 1}$, exactly one of α, β is in $\{0, 3\}$, and switching it between 0 and 3 always leads to a minus sign. Similar logic applies to R_{12} and R_{22}.

Then the trace of the Ricci tensor is zero, so $R_{\mu\nu} = G_{\mu\nu}$, and raising indices gives

$$G^{00} = G^{33} = -G^{03} = 4h_+\ddot{h}_+ + 2\dot{h}_+^2 + 4h_\times\ddot{h}_\times + 2\dot{h}_\times^2, \quad \text{other components zero.} \tag{13}$$

C. Effective stress-energy tensor

We are now ready to obtain the effective stress-energy tensor $t^{\mu\nu}$. Since the gravitational wave is a solution to the linearized Einstein equations in vacuum, we have $H^{\mu\alpha\nu\beta}{}_{,\alpha\beta} = 0$, and hence

$$t^{\mu\nu} = -\frac{1}{8\pi} G^{\mu\nu}. \tag{14}$$

That is,

$$t^{00} = t^{33} = -t^{03} = \frac{1}{4\pi}(2h_+\ddot{h}_+ + \dot{h}_+^2 + 2h_\times\ddot{h}_\times + \dot{h}_\times^2), \quad \text{other components zero.} \tag{15}$$

Now if we are interested in the average effective energy flux carried by gravitational waves, we may average this over a wave period. In fact, it is **not even meaningful** to average this over a fraction of a wave period, since the result is then gauge-dependent. The operation of measuring the mass of the central object using Gaussian integrals is only well-defined to first order in perturbation theory, so the periodic variations in "mass" over a wave period – which are second-order – cannot be meaningful. Moreover, the operation of using Kepler's third law, or even using the gravitational deflection of light, to measure mass, cannot be performed in the far field radiation zone in less than a wave period. These issues are connected with the fact that the spacetime structure does not locally have any energy at all: it locally looks like Minkowski space!

Therefore, we are interested in taking the average of $t^{\mu\nu}$ smoothed over a wave period. We note that by integration by parts,

$$\langle h_+\ddot{h}_+ + h_\times\ddot{h}_\times\rangle = -\langle \dot{h}_+\dot{h}_+ + \dot{h}_\times\dot{h}_\times\rangle. \tag{16}$$

Then:

$$\langle t^{00}\rangle = \langle t^{33}\rangle = -\langle t^{03}\rangle = \frac{1}{4\pi}\langle \dot{h}_+^2 + \dot{h}_\times^2\rangle, \quad \text{other components zero.} \tag{17}$$

This is equivalent to the stress-energy tensor of a beam of massless particles such as photons: the energy density, momentum density, and momentum flux are all equal, and all momentum and transport is in the 3-direction. It is for this reason that we can think of the gravitational waves as being the wave equivalent of a massless particle, the *graviton*.

The energy density above can be re-written in terms of the components of $h_{ij}^{\rm TT}$ in the transverse-traceless gauge:

$$\langle t^{00}\rangle = \frac{1}{32\pi}\langle h_{ij}^{\rm TT} h_{ij}^{\rm TT}\rangle. \tag{18}$$

IV. ENERGY LOSS DUE TO GRAVITATIONAL WAVE EMISSION

We are finally in a position to consider the energy loss of a system via emission of gravitational waves. In Lecture XII, we found that the transverse-traceless gauge gravitational wave emitted from a source was

$$h_{ij}^{\rm TT} = \frac{1}{R}\left(2\ddot{Q}_{ij} + n_kn_l\ddot{Q}_{kl}\delta_{ij} + n_in_jn_kn_l\ddot{Q}_{kl} - 2n_jn_k\ddot{Q}_{ik} - 2n_in_k\ddot{Q}_{jk}\right). \tag{19}$$

We now want to know the effective energy density in said gravitational waves far from the source. Using Eq. (18), and recalling that Q_{ij} is traceless, this is

$$\begin{aligned}
\langle t^{00}\rangle &= \frac{1}{32\pi R^2}\left(2\dddot{Q}_{ij} + n_kn_l\dddot{Q}_{kl}\delta_{ij} + n_in_jn_kn_l\dddot{Q}_{kl} - 2n_jn_k\dddot{Q}_{ik} - 2n_in_k\dddot{Q}_{jk}\right) \\
&\quad\times\left(2\dddot{Q}_{ij} + n_mn_n\dddot{Q}_{mn}\delta_{ij} + n_in_jn_mn_n\dddot{Q}_{mn} - 2n_jn_m\dddot{Q}_{im} - 2n_in_m\dddot{Q}_{jm}\right) \\
&= \frac{1}{32\pi R^2}\Big(4\dddot{Q}_{ij}\dddot{Q}_{ij} + 4\dddot{Q}_{ij}n_in_j\dddot{Q}_{mn}n_mn_n - 16\dddot{Q}_{ij}\dddot{Q}_{im}n_jn_m \\
&\quad -3\dddot{Q}_{ij}n_in_j\dddot{Q}_{mn}n_mn_n - 7\dddot{Q}_{ij}n_in_j\dddot{Q}_{mn}n_mn_n + 8\dddot{Q}_{ij}\dddot{Q}_{im}n_jn_m + 8\dddot{Q}_{ij}n_in_j\dddot{Q}_{mn}n_mn_n\Big) \\
&= \frac{1}{32\pi R^2}\left(4\dddot{Q}_{ij}\dddot{Q}_{ij} + 2\dddot{Q}_{ij}n_in_j\dddot{Q}_{mn}n_mn_n - 8\dddot{Q}_{ij}\dddot{Q}_{im}n_jn_m\right).
\end{aligned} \tag{20}$$

Of greatest interest to us is the total emitted power,

$$-\langle \dot{E} \rangle = \int_{\partial \mathcal{V}} \langle t^{0i} \rangle n_i \, d^2x = R^2 \int_{S^2} \langle t^{0i} \rangle n_i \, d^2n = R^2 \int_{S^2} \langle t^{00} \rangle \, d^2n. \tag{21}$$

To obtain this, we perform the integral of Eq. (20) over the whole sphere. We use the relations

$$\int_{S^2} d^2n = 4\pi, \quad \int_{S^2} n_i n_j \, d^2n = \frac{4\pi}{3}\delta_{ij} \quad \text{and} \quad \int_{S^2} n_i n_j n_k n_l \, d^2n = \frac{4\pi}{15}(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}). \tag{22}$$

Then we have

$$-\langle \dot{E} \rangle = \frac{1}{32\pi} \left\langle 16\pi \dddot{Q}_{ij} \dddot{Q}_{ij} + \frac{8\pi}{15} \dddot{Q}_{ij} \dddot{Q}_{mn} (\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}) - \frac{32\pi}{3} \dddot{Q}_{ij} \dddot{Q}_{im} \delta_{jm} \right\rangle. \tag{23}$$

This simplifies to

$$-\langle \dot{E} \rangle = \frac{1}{5} \langle \dddot{Q}_{ij} \dddot{Q}_{ij} \rangle, \tag{24}$$

a remarkably simple formula!

A. Angular momentum carried

At a finite distance from the source, the wavefronts are not exactly spherical. Therefore, the effective stress-energy tensor has nonradial components and there is a net "effective angular momentum density" $\epsilon_{ijk} R n^j t^{0k}$, where the non-radial part of t^{0k} scales as $\propto R^{-3}$. The calculation is very similar to the above but of course much more tedious since it carries the next-leading term in $1/R$. It leads to

$$-\langle \dot{S}_i \rangle = \frac{2}{5} \epsilon_{ijk} \langle \ddot{Q}_{lj} \dddot{Q}_{lk} \rangle. \tag{25}$$