Lecture XIII: Weakly nonlinear GR in asymptotically flat spacetime – Concepts

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I. OVERVIEW

The previous discussions have taken place in the context of linearized GR, which is not a fully consistent theory. We will now discuss some aspects of GR in the nonlinear regime, with particular attention to isolated systems. Our eventual goal is to compute the energy loss of a system emitting gravitational waves. We will outline the concepts here, and examine the calculation in the next lecture.

The recommended reading for this lecture is:

- MTW §19.3, 20.1–20.3.

II. MASS AND ANGULAR MOMENTUM OF AN ISOLATED SYSTEM

In the previous discussion on linearized, we showed that the metric at large distances from an isolated system in linearized gravity could be written as

$$ds^2 = -\left(1-2\frac{M}{R}\right)dt^2 + 4\epsilon_{ijk}\frac{n_j S_k}{R^2}\,dx^i\,dt + \left(1+2\frac{M}{R}\right)[(dx^1)^2+(dx^2)^2+(dx^3)^2] + [\text{gravitational wave terms}], \quad (1)$$

where the gravitational wave terms decay as $\sim 1/R$.

A. Pseudotensors and Gaussian integrals

Let us consider the object

$$H^{\mu\alpha\nu\beta} \equiv -\bar h^{\mu\nu}\eta^{\alpha\beta} - \eta^{\mu\nu}\bar h^{\alpha\beta} + \bar h^{\alpha\nu}\eta^{\mu\beta} + \bar h^{\mu\beta}\eta^{\alpha\nu}. \quad (2)$$

As this is a perturbation on a Minkowski background, we will raise and lower indices on $H^{\mu\alpha\nu\beta}$ with respect to $\eta^{\mu\nu}$.

This object satisfies the following symmetry properties:

$$\begin{aligned}
H^{(\mu\alpha)\nu\beta} &= 0 \quad \text{antisymmetric on first two indices,}\\
H^{\mu\alpha(\nu\beta)} &= 0 \quad \text{antisymmetric on last two indices,}\\
H^{\mu\alpha\nu\beta} &= H^{\nu\beta\mu\alpha} \quad \text{symmetric under interchange of first and last two indices, and}\\
H^{\mu[\alpha\nu\beta]} &= 0 \quad \text{Jacobi identity.}
\end{aligned} \quad (3)$$

So far these statements do *not* depend on linear perturbation theory; they are just definitions. In linear theory, however, we have

$$H^{\mu\alpha\nu\beta}{}_{,\alpha\beta} = 2G^{\mu\nu} = 16\pi T^{\mu\nu}. \quad (4)$$

This enables us to write some integrals for the total momentum in a volume $\mathcal{V}$:

$$P^\mu = \int_{\mathcal{V}} T^{\mu 0}\,d^3x$$

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$$
\begin{aligned}
&= \frac{1}{16\pi}\int_{\mathcal{V}} H^{\mu\alpha 0\beta}{}_{,\alpha\beta}\, d^3x \\
&= \frac{1}{16\pi}\int_{\mathcal{V}} H^{\mu\alpha 0j}{}_{,\alpha j}\, d^3x \\
&= -\frac{1}{16\pi}\int_{\partial\mathcal{V}} H^{\mu\alpha 0j}{}_{,\alpha} n_j\, d^2x
\end{aligned} \tag{5}
$$

In particular, the total enclosed energy is

$$
\begin{aligned}
E = P^0 &= -\frac{1}{16\pi}\int_{\partial\mathcal{V}} H^{0i0j}{}_{,i} n_j\, d^2x \\
&= -\frac{1}{16\pi}\int_{\partial V} (-\bar{h}^{00}\delta^{ij} + \bar{h}^{ij})_{,i} n_j\, d^2x \\
&= \frac{1}{16\pi}\int_{\partial V} (\bar{h}^{00}{}_{,i} n_i - \bar{h}^{ij}{}_{,i} n_j)\, d^2x.
\end{aligned} \tag{6}
$$

We may also determine the mass dipole moment:

$$
\begin{aligned}
\Delta^i &= \int_{\mathcal{V}} x^i T^{00}\, d^3x \\
&= \frac{1}{16\pi}\int_{\mathcal{V}} x^i H^{0j0k}{}_{,jk}\, d^3x \\
&= \frac{1}{16\pi}\left[\int_{\partial\mathcal{V}} x^i H^{0j0k}{}_{,k} n_j\, d^2x - \int_{\mathcal{V}} H^{0i0k}{}_{,k}\, d^3x\right] \\
&= \frac{1}{16\pi}\left[\int_{\partial\mathcal{V}} x^i H^{0j0k}{}_{,k} n_j\, d^2x + \int_{\partial\mathcal{V}} H^{0i0k} n_k\, d^3x\right].
\end{aligned} \tag{7}
$$

We may also consider the enclosed angular momentum:

$$
\begin{aligned}
S_i &= \epsilon_{ijk}\int_{\mathcal{V}} x^j T^{0k}\, d^3x \\
&= \frac{1}{16\pi}\epsilon_{ijk}\int_{\mathcal{V}} x^j H^{0\alpha k\beta}{}_{,\alpha\beta}\, d^3x \\
&= \frac{1}{16\pi}\epsilon_{ijk}\int_{\mathcal{V}} x^j H^{0\alpha k\beta}{}_{,\alpha\beta}\, d^3x \\
&= \frac{1}{16\pi}\epsilon_{ijk}\int_{\mathcal{V}} x^j H^{0lk\beta}{}_{,l\beta}\, d^3x \\
&= \frac{1}{16\pi}\epsilon_{ijk}\left[\int_{\partial\mathcal{V}} x^j H^{0lk\beta}{}_{,\beta} n_l\, d^2x - \int_{\mathcal{V}} H^{0jk\beta}{}_{,\beta}\, d^3x\right].
\end{aligned} \tag{8}
$$

Now H^{0jk0} is symmetric in j and k, so in the last integral we may replace $H^{0jk\beta}{}_{,\beta} \to H^{0jkl}{}_{,l}$. Then:

$$
S_i = \frac{1}{16\pi}\epsilon_{ijk}\left[\int_{\partial\mathcal{V}} x^j H^{0lk\beta}{}_{,\beta} n_l\, d^2x + \int_{\partial\mathcal{V}} H^{0jkl} n_l\, d^2x\right]. \tag{9}
$$

B. Application to asymptotically flat spacetimes

In the limit of an asymptotically flat spacetime, it should be permissible to use the surface integrals, Eqs. (5,6,7,9) far from the source. It is therefore possible to speak of the *mass*, *momentum*, *center of mass*, and *angular momentum* of such a source, even if it contains strong gravitational fields, or even if it is a black hole.

In such situations, it is useful to define the effective stress-energy pseudotensor $T^{\mathrm{eff}\,\mu\nu}$ by

$$
T^{\mathrm{eff}\,\mu\nu} \equiv \frac{1}{16\pi} H^{\mu\alpha\nu\beta}{}_{,\alpha\beta}. \tag{10}
$$

In linearized GR, this is exactly equal to the stress-energy tensor. In full GR, it is not: the difference is called the *gravitational stress-energy psuedotensor* $t^{\mu\nu}$:

$$t^{\mu\nu} \equiv T^{\text{eff}\,\mu\nu} - T^{\mu\nu}. \tag{11}$$

Note that $t^{\mu\nu}$ is *not* a tensor: under general coordinate transformations, it has no reason to be well-behaved. However, it does transform in the usual way under **global** Lorentz transformations of the background.

Warning: Once again, $T^{\text{eff}\,\mu\nu}$ and $t^{\mu\nu}$ are not tensors. It is **not** possible to take the energy and angular momentum of an object with strong gravity and "localize" the part associated with the gravitational field; such a description cannot be gauge-invariant. However, the overall integrals are gauge-invariant.

The antisymmetry relations imply that

$$T^{\text{eff}\,\mu\nu}{}_{,\nu} = \frac{1}{16\pi} H^{\mu\alpha\nu\beta}{}_{,\alpha\beta\nu} = 0. \tag{12}$$

This is a sort of conservation law for the effective stress-energy pseudotensor. It is distinct from the law obeyed by the true stress-energy tensor $T^{\mu\nu}{}_{;\nu} = 0$.

The point of the stress-energy pseudotensor is that even for sources containing strong gravitational fields, it integrates to the proper momentum, center of mass, and angular momentum. For example, the relation

$$P^{\mu} = \int_{\mathcal{V}} T^{\text{eff}\,\mu 0}\, d^3x \tag{13}$$

where P^{μ} is defined for a self-gravitating object by Eq. (5) remains valid even for a neutron star. The conservation law, Eq. (12) then implies the usual relations:

- For an isolated system with no emerging gravitational radiation, P^{μ} is conserved.
- For an isolated system with no emerging gravitational radiation, the rate of change of the mass dipole moment $\dot{\Delta}^i = P^i$.
- For an isolated system with no emerging gravitational radiation, S_i is conserved.

By "no emerging gravitational radiation" we mean to set $t^{\mu\nu}$ to be negligible at large distances (see below).

We haven't proven these rules for a black hole or other system with "weird" topology (such that you can't actually do an integral of T^{eff} through the center of the object). Nevertheless the integral definitions combined with the equivalent conservation rule $H^{\mu\alpha\nu\beta}{}_{,\alpha\beta\nu}$ are sufficient to prove this. (Homework exercise!)

For a system emitting gravitational radiation, the situation is more complex. The Einstein tensor is now given not by $\frac{1}{2}H^{\mu\alpha\nu\beta}{}_{,\alpha\beta}$, but by higher-order terms as well:

$$G^{\mu\nu} = \frac{1}{2} H^{\mu\alpha\nu\beta}{}_{,\alpha\beta} - \frac{1}{8\pi} t^{\mu\nu}, \tag{14}$$

where $t^{\mu\nu}$ contains all terms 2^{nd} order and higher in $\bar{h}^{\mu\nu}$. The amplitude of emitted gravitational waves is $\propto 1/R$ and hence the gravitational stress-energy psuedotensor, being a second-order object, is $\propto 1/R^2$. This is important because if we write the time derivative of e.g. the system's energy, and assume no matter is emitted ($T^{\mu\nu} = 0$ on the boundary of the region considered)

$$\dot{E} = \int_{\mathcal{V}} \dot{T}^{\text{eff}\,00}\, d^3x = -\int_{\mathcal{V}} T^{\text{eff}\,0i} n_i\, d^3x = -\int_{\partial\mathcal{V}} T^{\text{eff}\,0i} n_i\, d^2x = -\int_{\partial\mathcal{V}} t^{0i} n_i\, d^2x. \tag{15}$$

For the case of emerging gravitational radiation, the latter integral approaches a constant as we take the surrounding surface to ∞. Therefore a system can change its total energy through the emission of such waves. Similar rules tell us that it can emit angular momentum.

In order to understand the quantitative implications of all this, we need to develop the formula for $t^{\mu\nu}$, and determine the "effective" energy carried by gravitational waves. The computation of $t^{\mu\nu}$ to second order will be our next order of business.