Lecture XII: External field of an isolated system

Christopher M. Hirata

Caltech M/C 350-17, Pasadena CA 91125, USA[*]

(Dated: November 16, 2011)

I. OVERVIEW

Having examined weak field gravity and the associated experimental tests, we now turn our attention to the external field produced by a system (e.g. a star or the solar system) in linearized gravity. This will include both the gravitational analogues of electric and magnetic multipole moments, as well as gravitational waves.

The discussion in the beginning part of this lecture makes sense only for weak perturbations around Minkowski spacetime. Later we will generalize the concepts to make sense in *asymptotically flat* spacetime – i.e. spacetime that looks like Minkowski far from the system, but may have strongly curved regions inside of it (e.g. a black hole).

The recommended reading for this lecture is:

- MTW §19.1–19.2.

II. GREEN'S FUNCTION SOLUTION FOR THE METRIC PERTURBATION

We found that in Lorentz gauge the trace-reversed metric perturbation is given via the relation

$$\Box\bar h_{\mu\nu} = -16\pi T_{\mu\nu}. \tag{1}$$

We would like to formally solve this equation using a Green's function approach. That is, we wish to construct the *Green's function* $G(x^\alpha)$ such that

$$\Box G(x^\alpha) = \delta^{(4)}(x^\alpha), \tag{2}$$

and then by the principle of superposition we may write the metric perturbation as an integral over the Green's function:

$$\bar h_{\mu\nu}(x^\alpha) = -16\pi \int G(x^\alpha - y^\alpha) T_{\mu\nu}(y^\alpha)\, d^4\boldsymbol{y}. \tag{3}$$

How are we to find G? There is unfortunately no unique answer! After all, we could add any function f with $\Box f = 0$ to G and it would still satisfy Eq. (2). However, in most situations there is a physical choice: we want the *retarded Green's function* $G_{\rm ret}$, which is zero for $x^0 = t < 0$. This corresponds to the solution in which there is no incoming gravitational radiation. The use of the retarded Green's function is however not a necessity but a particular solution to Einstein's equations.

The easiest method to obtain the retarded Green's function is to take the Fourier transform of Eq. (2) over all four dimensions. Defining

$$\tilde G(k_\alpha) = \int G(x^\alpha)\, e^{-ik_\alpha x^\alpha}\, d^4\boldsymbol{x}, \tag{4}$$

we find that the Fourier transform of Eq. (2) is

$$\int [\Box G(x^\alpha)]\, e^{-ik_\alpha x^\alpha}\, d^4x = \int \delta^{(4)}(x^\alpha)\, e^{-ik_\alpha x^\alpha}\, d^4\boldsymbol{x}. \tag{5}$$

[*]Electronic address: `chirata@tapir.caltech.edu`

Since $\Box = \partial_\mu \partial^\mu$, we may use double integration by parts on the left-hand side to move the $\Box$ onto the complex exponential. On the right-hand side, the integral is simply 1:

$$\int G(x^\alpha)\,\Box e^{-ik_\alpha x^\alpha}\, d^4\boldsymbol{x} = 1. \tag{6}$$

The box acting on the complex exponential brings down a factor of $-k_\mu k^\mu$, so

$$-k_\mu k^\mu \tilde{G}_{\rm ret}(k_\alpha) = 1, \tag{7}$$

or

$$\tilde{G}(k_\alpha) = -\frac{1}{k_\mu k^\mu} = \frac{1}{(k_0)^2 - k_{\rm sp}^2}, \tag{8}$$

where the "sp" subscript denotes the spacelike part.

As usual with Fourier transforms, we may recover the original function by writing

$$G(x^\alpha) = (2\pi)^{-4} \int \tilde{G}(k_\alpha) e^{ik_\alpha x^\alpha}\, d^4\boldsymbol{k}. \tag{9}$$

There's a big problem with this integral as it stands, which is that the integrand diverges when $k_\mu k^\mu = 0$. The resolution to this problem is to appeal to the retarded condition: since $G(x^\alpha) = 0$ for $t < 0$, it follows that the integral in Eq. (4) is well-behaved if we add a small negative imaginary part to k_0, since taking $k_0 \to k_0 - i\epsilon$ introduces a factor of $e^{-\epsilon t}$ into the integrand. If we make this choice – which essentially determines the "retarded" nature of the Green's function – we find

$$G_{\rm ret}(x^\alpha) = (2\pi)^{-4} \int \frac{1}{(k_0 - i\epsilon)^2 - k_{\rm sp}^2} e^{i\boldsymbol{k}_{\rm sp}\cdot\boldsymbol{x}_{\rm sp}} e^{ik_0 t}\, d^4\boldsymbol{k}. \tag{10}$$

The k_0 part of the integral is solvable by contour integration. In particular, we define the integral

$$I(k_{\rm sp}, t) = \int_{-\infty}^{\infty} \frac{1}{(k_0 - i\epsilon)^2 - k_{\rm sp}^2} e^{ik_0 t}\, dk_0. \tag{11}$$

The integrand is analytic except for poles at $k_0 = \pm k_{\rm sp} + i\epsilon$. If $t < 0$, then the integrand goes to zero in the lower half complex plane, and we may close the contour and find an integral of zero. If $t > 0$, then the integrand goes to zero in the upper half complex plane:

$$I(k_{\rm sp}, t) = 2\pi i \left[\frac{e^{-ik_{\rm sp}t}}{-2k_{\rm sp}} + \frac{e^{ik_{\rm sp}t}}{2k_{\rm sp}}\right] = -2\pi \frac{\sin k_{\rm sp} t}{k_{\rm sp}}. \tag{12}$$

It then follows that:

$$G_{\rm ret}(x^\alpha) = -(2\pi)^{-3}\Theta(t) \int \frac{\sin k_{\rm sp} t}{k_{\rm sp}} e^{i\boldsymbol{k}_{\rm sp}\cdot\boldsymbol{x}_{\rm sp}}\, d^3\boldsymbol{k}_{\rm sp}, \tag{13}$$

where Θ is the step function. We may further simplify the integral by choosing a coordinate system where the 3-axis is along $\boldsymbol{x}_{\rm sp}$, and define spherical coordinates for the wave vector $\boldsymbol{k}_{\rm sp}$: $(k_{\rm sp}, \mu = \cos\theta, \phi)$. The integral then becomes

$$G_{\rm ret}(x^\alpha) = -(2\pi)^{-3}\Theta(t) \int \frac{\sin k_{\rm sp} t}{k_{\rm sp}} e^{ik_{\rm sp} x_{\rm sp}\mu}\, k_{\rm sp}^2\, dk_{\rm sp}\, d\mu\, d\phi. \tag{14}$$

The ϕ integral simply evaluates to 2π, and the μ integral is trivial:

$$\int_{-1}^{1} e^{ik_{\rm sp} x_{\rm sp}\mu}\, d\mu = 2\frac{\sin k_{\rm sp} x_{\rm sp}}{k_{\rm sp} x_{\rm sp}}. \tag{15}$$

Thus:

$$G_{\rm ret}(x^\alpha) = \frac{1}{2\pi^2 x_{\rm sp}} \Theta(t) \int_0^\infty \sin(k_{\rm sp} t) \sin(k_{\rm sp} x_{\rm sp})\, dk_{\rm sp}. \tag{16}$$

If we use product-to-sum identities:

$$G_{\rm ret}(x^\alpha) = -\frac{1}{4\pi^2 x_{\rm sp}}\Theta(t)\int_0^\infty \{\cos[k_{\rm sp}(t-x_{\rm sp})] - \cos[k_{\rm sp}(t+x_{\rm sp})]\}\, dk_{\rm sp}. \tag{17}$$

If $t \approx x_{\rm sp}$ then this is ill-behaved. We need to use the following form of the δ-function:

$$\int_0^\infty \cos ku\, dk = \pi\delta(u). \tag{18}$$

Then

$$G_{\rm ret}(x^\alpha) = -\frac{1}{4\pi x_{\rm sp}}\Theta(t)\delta(t - x_{\rm sp}). \tag{19}$$

This is the form of the retarded Green's function that we will use.

The retarded solution for the metric is then

$$\Box \bar{h}^{\rm ret}_{\mu\nu}(\boldsymbol{x}_{\rm sp}, t) = 4\int \frac{1}{r} T_{\mu\nu}(\boldsymbol{y}_{\rm sp}, t-r) d^3\boldsymbol{y}_{\rm sp}, \tag{20}$$

where $r \equiv |\boldsymbol{x}_{\rm sp} - \boldsymbol{y}_{\rm sp}|$.

Remember that the retarded solution is only one possible solution for the metric for a given matter source. But the difference between the true solution and the retarded solution can be found by noting that if both are soutions of $\Box\bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}$ for the same source, then

$$\Box(\bar{h}_{\mu\nu} - \bar{h}^{\rm ret}_{\mu\nu}) = 0. \tag{21}$$

Therefore, we may write

$$\bar{h}_{\mu\nu} = \bar{h}^{\rm ret}_{\mu\nu} + \bar{h}^{\rm homo}_{\mu\nu}, \tag{22}$$

where $\bar{h}^{\rm homo}_{\mu\nu}$ is a homogeneous or gravitational wave solution (solution for zero matter source). We will see that Eq. (20) contains gravitational waves, but they are only outgoing (as one can see from the $t - r$ time argument). Therefore setting $\bar{h}^{\rm homo}_{\mu\nu}$ to zero is equivalent to saying that no external gravitational waves are incident on a system. Usually this is a good approximation!

[**Warning**: The outgoing gravitational waves in the above formulation are not necessarily in transverse-traceless gauge.]

III. MULTIPOLE EXPANSION

Far from a source, it is common to do a *multipole expansion*: essentially a power-series expansion of Eq. (20) in powers of $\boldsymbol{y}_{\rm sp}$. This can be done for both relativistic and nonrelativistic sources, but we will focus on nonrelativistic sources (e.g. binary stars) here. We will also examine only the lowest-order multipoles since these (i) correspond to conserved quantities and (ii) the next-lowest terms carry the dominant source of gravitational radiation from a nonrelativistic object.

A. The trace-reversed perturbation

We begin by expanding $1/r$ as a power series in y:

$$\begin{aligned}
\frac{1}{r} &= [(x_i - y_i)(x_i - y_i)]^{-1/2} \\
&= [x_i x_i - 2x_i y_i + y_i y_i]^{-1/2} \\
&= (x_i x_i)^{-1/2} - \frac{1}{2}(x_i x_i)^{-3/2}(-2x_i y_i + y_i y_i) + \frac{3}{8}(x_i x_i)^{-5/2}(2x_i y_i)^2 + ... \\
&= \frac{1}{|\boldsymbol{x}_{\rm sp}|} - \frac{x_i y_i}{|\boldsymbol{x}_{\rm sp}|^3} + \frac{x_i x_j(3y_i y_j - y_k y_k \delta_{ij})}{2|\boldsymbol{x}_{\rm sp}|^5} + ...,
\end{aligned} \tag{23}$$

with a fractional error of order $(L/x_{\rm sp})^3$, where L is the typical size scale of the source.

We may also Taylor-expand $T_{\mu\nu}$:

$$T_{\mu\nu}(\boldsymbol{y}_{\rm sp}, t-r) = T_{\mu\nu}(\boldsymbol{y}_{\rm sp}, t-|\boldsymbol{x}_{\rm sp}|) + \frac{x_i y_i}{|\boldsymbol{x}_{\rm sp}|} T_{\mu\nu,0}(\boldsymbol{y}_{\rm sp}, t-|\boldsymbol{x}_{\rm sp}|) + \frac{x_i x_j y_i y_j}{2|\boldsymbol{x}_{\rm sp}|^2} T_{\mu\nu,00}(\boldsymbol{y}_{\rm sp}, t-|\boldsymbol{x}_{\rm sp}|)..., \tag{24}$$

with a fractional error of order $L^3/t^3 = V^3$ or $L^2/x_{\rm sp}t = VL/x_{\rm sp}$, where V is the typical velocity scale of the source.

It then follows that the retarded solution for the metric perturbation is

$$\begin{aligned} \bar{h}_{\mu\nu}(\boldsymbol{x}_{\rm sp}, t) &= 4\int \left[\frac{1}{R} - \frac{x_i y_i}{R^3} + \frac{x_i x_j(3y_i y_j - y_k y_k \delta_{ij})}{2R^5}...\right] \\ &\quad \times \left[T_{\mu\nu}(\boldsymbol{y}_{\rm sp}, t-R) + \frac{x_i y_i}{R}\dot{T}_{\mu\nu}(\boldsymbol{y}_{\rm sp}, t-R) + \frac{x_i x_j y_i y_j}{2|\boldsymbol{x}_{\rm sp}|^2} T_{\mu\nu,00}(\boldsymbol{y}_{\rm sp}, t-|\boldsymbol{x}_{\rm sp}|)\right] d^3\boldsymbol{y}_{\rm sp}, \end{aligned} \tag{25}$$

where we define $R = |\boldsymbol{x}_{\rm sp}|$. The lowest-order fractional errors are V^2, VL/R, and $1/R^3$.

It is useful to consider the behavior of each metric component at large distances from the source. We first see that

$$\begin{aligned} \frac{1}{4}\bar{h}_{00}(\boldsymbol{x}_{\rm sp}, t) &= \frac{1}{R}\int \rho\, d^3\boldsymbol{y}_{\rm sp} - \frac{x_i}{R^3}\int y_i \rho\, d^3\boldsymbol{y}_{\rm sp} + \frac{x_i}{R^2}\int y_i\dot{\rho}\, d^3\boldsymbol{y}_{\rm sp} + \frac{3x_i x_j}{2R^5}\int (y_i y_j - \frac{1}{3}y_k y_k \delta_{ij})\rho\, d^3\boldsymbol{y}_{\rm sp} \\ &\quad + \frac{x_i x_j}{2R^2}\int y_i y_j \ddot{\rho}\, d^3\boldsymbol{y}_{\rm sp} + ...\bigg|_{\rm ret}, \end{aligned} \tag{26}$$

where the $|_{\rm ret}$ on the right-hand side indicates evaluation at the retarded time $t-R$.

Now the first integral is simply the (conserved) mass M of the system. (Technically this is the energy, but the difference only arises at the next order in velocity, which we have dropped.) The second integral is the mass dipole moment,

$$MY_i = \int y_i \rho\, d^3\boldsymbol{y}_{\rm sp}, \tag{27}$$

where Y_i is the center of mass. Finally, defining the momentum density $F_j = T^0{}_j$ and noting that $F_{j,j} = -\dot{\rho}$ the third integral is

$$M\dot{Y}_i = \int y_i\dot{\rho}\, d^3\boldsymbol{y}_{\rm sp} = -\int y_i F_{j,j}\, d^3\boldsymbol{y}_{\rm sp} = \int y_{i,j}F_j\, d^3\boldsymbol{y}_{\rm sp} = \int F_i\, d^3\boldsymbol{y}_{\rm sp} = P_i, \tag{28}$$

where P_i is the total momentum. Under normal circumstances, we will work in the *center of mass frame*, in which Y_i and P_i are zero. Finally, we define the mass quadrupole moment of the system to be

$$Q_{ij} = \int (y_i y_j - \frac{1}{3}y_k y_k \delta_{ij})\rho\, d^3\boldsymbol{y}_{\rm sp} = I_{ij} - \frac{1}{3}I_{kk}\delta_{ij}. \tag{29}$$

(Here I_{ij} is the moment of inertia tensor.) Then Eq. (25) reduces to

$$\bar{h}_{00}(\boldsymbol{x}) = 4\frac{M}{R} + 6\frac{Q_{ij}x_i x_j}{R^5} + 2\frac{x_i x_j}{R^3}\ddot{I}_{ij}...\bigg|_{\rm ret}. \tag{30}$$

This is simply the standard quadrupolar formula familiar from Newtonian physics. **But** note that in fully nonlinear general relativity, the leading correction to the M/R formula is not the quadrupole term ($\propto R^{-3}$) but rather the nonlinearity of the theory ($\propto M^2/R^2$ – this is the term that is necessary to obtain the perihelion precession of Mercury). In some cases, such as the orbit of Mercury around the Sun, the relativistic M^2/R^2 term dominates over the quadrupole effect, but for a satellite orbiting the Earth (with its much more flattened shape) the opposite is true.

We may further write down the linearized formula for $\bar{h}_{0i}$, including terms through second order in V:

$$\bar{h}_{0i}(\boldsymbol{x}) = \frac{4}{R}\int F_i(\boldsymbol{y}_{\rm sp})\, d^3\boldsymbol{y}_{\rm sp} - 4\frac{x_j}{R^3}\int y_j F_i(\boldsymbol{y}_{\rm sp})\, d^3\boldsymbol{y}_{\rm sp} + 4\frac{x_j}{R^2}\int y_j\dot{F}_i\, d^3\boldsymbol{y}_{\rm sp} + ...\bigg|_{\rm ret}; \tag{31}$$

the first integral is P_i and hence vanishes. The second integral may be simplified using the rule that the angular momentum is

$$S_k = -\epsilon_{ijk}\int y_j F_i(\boldsymbol{y}_{\rm sp})\, d^3\boldsymbol{y}_{\rm sp} \tag{32}$$

so that

$$\epsilon_{klm}S_k = -\epsilon_{klm}\epsilon_{ijk}\int y_j F_i(\boldsymbol{y}_{\rm sp})\, d^3\boldsymbol{y}_{\rm sp} = -(\delta_{li}\delta_{mj} - \delta_{lj}\delta_{mi})\int y_j F_i(\boldsymbol{y}_{\rm sp})\, d^3\boldsymbol{y}_{\rm sp} = -2\int y_{[m}F_{l]}(\boldsymbol{y}_{\rm sp})\, d^3\boldsymbol{y}_{\rm sp}. \tag{33}$$

Furthermore,

$$\dot{I}_{ij} = \int y_i y_j \dot{\rho}\, d^3\boldsymbol{y}_{\rm sp} = -\int y_i y_j F_{k,k}\, d^3\boldsymbol{y}_{\rm sp} = \int (y_{i,k}y_j F_k + y_{j,k}y_i F_k)\, d^3\boldsymbol{y}_{\rm sp} = 2\int y_{(j}F_{i)}\, d^3\boldsymbol{y}_{\rm sp}. \tag{34}$$

Therefore, decomposing the integral in $\bar{h}_{0i}$ into its symmetric and antisymmetric parts:

$$-4\frac{x_j}{R^3}\int y_j F_i(\boldsymbol{y}_{\rm sp})\, d^3\boldsymbol{y}_{\rm sp} = -4\frac{x_j}{R^3}\left[\int y_{[j}F_{i]}(\boldsymbol{y}_{\rm sp})\, d^3\boldsymbol{y}_{\rm sp} + \int y_{(j}F_{i)}(\boldsymbol{y}_{\rm sp})\, d^3\boldsymbol{y}_{\rm sp}\right] = 2\epsilon_{ijk}S_k\frac{x_j}{R^3} + 2\dot{I}_{ij}\frac{x_j}{R^3}. \tag{35}$$

Substituting in this, and its time derivative, gives

$$\bar{h}_{0i}(\boldsymbol{x}) = 2\epsilon_{ijk}S_k\frac{x_j}{R^3} + 2\dot{I}_{ij}\frac{x_j}{R^3} - 2\epsilon_{ijk}\dot{S}_k\frac{x_j}{R^2} - 2\ddot{I}_{ij}\frac{x_j}{R^2} + ...\Big|_{\rm ret}. \tag{36}$$

Using conservation of angular momentum, we drop the $\dot{S}_k$ term, leaving

$$\bar{h}_{0i}(\boldsymbol{x}) = 2\epsilon_{ijk}S_k\frac{x_j}{R^3} + 2\dot{I}_{ij}\frac{x_j}{R^2} - 2\ddot{I}_{ij}\frac{x_j}{R^2} + ...\Big|_{\rm ret}. \tag{37}$$

Finally, in the case of the space-space components we will write only the lowest term,

$$\bar{h}_{ij}(\boldsymbol{x}) = \frac{4}{R}\int T_{ij}\, d^3\boldsymbol{y}_{\rm sp}\Big|_{\rm ret}. \tag{38}$$

This can be simplified by noting that from Eq. (34):

$$\ddot{I}_{ij} = 2\int y_{(j}\dot{F}_{i)}\, d^3\boldsymbol{y}_{\rm sp} = -2\int y_{(j}T_{i)k,k}\, d^3\boldsymbol{y}_{\rm sp} = 2\int T_{(i}y_{j),k}\, d^3\boldsymbol{y}_{\rm sp} = 2\int T_{ij}\, d^3\boldsymbol{y}_{\rm sp}. \tag{39}$$

Then:

$$\bar{h}_{ij}(\boldsymbol{x}) = \frac{2}{R}\ddot{I}_{ij} + ...\Big|_{\rm ret}. \tag{40}$$

The far-field metric perturbation from a source according to the above equations can be found by reversing the trace of $\bar{h}$. Letting $n_i = x_i/R$, this is:

$$\bar{h} = -4\frac{M}{R} + \frac{2}{R}\ddot{I}_{kk} - \frac{2}{R}n_i n_j\ddot{I}_{ij} + 6\frac{Q_{ij}n_i n_j}{R^3} + ...\Big|_{\rm ret}. \tag{41}$$

Including this with trace reversal gives

$$\begin{aligned}
h_{00} &= 2\frac{M}{R} + \frac{1}{R}\ddot{I}_{kk} + \frac{1}{R}n_i n_j\ddot{I}_{ij} + 3\frac{Q_{ij}n_i n_j}{R^3} + ...\Big|_{\rm ret} \\
h_{0i} &= 2\epsilon_{ijk}S_k\frac{n_j}{R^2} + 2\dot{I}_{ij}\frac{n_j}{R^2} - 2\frac{n_j}{R}\ddot{I}_{ij} + ...\Big|_{\rm ret} \quad \text{and} \\
h_{ij} &= 2\frac{M}{R}\delta_{ij} + \frac{2}{R}\ddot{I}_{ij} - \frac{1}{R}\ddot{I}_{kk}\delta_{ij} + \frac{1}{R}n_k n_l\ddot{I}_{kl}\delta_{ij} - 3\frac{Q_{ij}n_i n_j}{R^3} + ...\Big|_{\rm ret}.
\end{aligned} \tag{42}$$

This looks at first glance to have several pieces: there is the familiar Newtonian potential; there is a piece associated with the angular momentum in the time-space part (*gravitomagnetism*); and there is a set of outward-propagating waves (gravitational waves!) associated with time variation of the quadrupole moment.

B. The radiation terms

You will note that the gravitational waves ($\dot{I}_{ij}$ terms) in the far field do not look like the ones we studied: in particular they have time-space and time-time components and are not orthogonal to n_i (the outward direction). At large distances from the source, the leading-order ($1/R$) wave terms are

$$\begin{aligned} h_{00}^{\rm GW} &= \frac{1}{R}\ddot{I}_{kk} + \frac{1}{R}n_i n_j \ddot{I}_{ij}, \\ h_{0i}^{\rm GW} &= -2\frac{n_j}{R}\ddot{I}_{ij}, \quad \text{and} \\ h_{ij}^{\rm GW} &= \frac{2}{R}\ddot{I}_{ij} - \frac{1}{R}\ddot{I}_{kk}\delta_{ij} + \frac{1}{R}n_k n_l \ddot{I}_{kl}\delta_{ij} \end{aligned} \tag{43}$$

(all retarded). Fortunately, a gauge transformation can be used to eliminate most of these terms. Recall that under a change of coordinates ξ^μ, the metric tensor changed by $\Delta h_{\mu\nu} = -\xi_{\mu,\nu} - \xi_{\nu,\mu}$. We first perform a change of the time coordinate t,

$$\xi_0 = \frac{1}{2R}\dot{I}_{kk} + \frac{1}{2R}n_i n_j \dot{I}_{ij}, \tag{44}$$

leaving the spatial coordinates fixed ($\xi_i = 0$). Then the time-time component changes by

$$\Delta h_{00} = -2\dot{\xi}_0 = -\frac{1}{R}\ddot{I}_{kk} - \frac{1}{R}n_i n_j \ddot{I}_{ij}. \tag{45}$$

To find the correction $\Delta h_{0i} = -\xi_{0,i}$, we recall that since $\dot{I}_{ij}$ is evaluated at the retarded time, its spatial derivative contains a time derivative:

$$\partial_i I_{kl} = \dot{I}_{kl}\partial_i(t-R) = -\dot{I}_{kl}\partial_i R = -\dot{I}_{kl}n_i. \tag{46}$$

Thus:

$$\Delta h_{0i} = \frac{1}{2R}n_i\ddot{I}_{kk} + \frac{1}{2R}n_i n_j n_k \ddot{I}_{jk}. \tag{47}$$

After applying this transformation, the outgoing wave is

$$\begin{aligned} h_{00}^{\rm GW} &= 0, \\ h_{0i}^{\rm GW} &= \frac{1}{2R}n_i\ddot{I}_{kk} + \frac{1}{2R}n_i n_j n_k \ddot{I}_{jk} - 2\frac{n_j}{R}\ddot{I}_{ij}, \quad \text{and} \\ h_{ij}^{\rm GW} &= \frac{2}{R}\ddot{I}_{ij} - \frac{1}{R}\ddot{I}_{kk}\delta_{ij} + \frac{1}{R}n_k n_l \ddot{I}_{kl}\delta_{ij} \end{aligned} \tag{48}$$

(again retarded).

Finally, we may introduce a gauge transformation in the spatial coordinates to eliminate $h_{0i}^{\rm GW}$. We want $\Delta h_{0i} = -\dot{\xi}_i$ to cancel $h_{0i}^{\rm GW}$, so we choose ξ_i to be the time-integral of $h_{0i}^{\rm GW}$:

$$\xi_i = \frac{1}{2R}n_i\dot{I}_{kk} + \frac{1}{2R}n_i n_j n_k \dot{I}_{jk} - 2\frac{n_j}{R}\dot{I}_{ij}. \tag{49}$$

Now we find that the change to the purely spatial metric is

$$\begin{aligned} \Delta h_{ij} &= -\xi_{i,j} - \xi_{j,i} \\ &= n_j\dot{\xi}_i + n_i\dot{\xi}_j \\ &= \frac{1}{R}n_i n_j \ddot{I}_{kk} + \frac{1}{R}n_i n_j n_k n_l \ddot{I}_{kl} - \frac{2}{R}n_j n_k \ddot{I}_{ik} - \frac{2}{R}n_i n_k \ddot{I}_{jk}. \end{aligned} \tag{50}$$

The overall amplitude of the outgoing gravitational wave is then

$$\begin{aligned} h_{00}^{\rm GW} &= 0, \\ h_{0i}^{\rm GW} &= 0, \quad \text{and} \\ h_{ij}^{\rm GW} &= \frac{1}{R}\left(2\ddot{I}_{ij} - \ddot{I}_{kk}\delta_{ij} + n_k n_l \ddot{I}_{kl}\delta_{ij} + n_i n_j \ddot{I}_{kk} + n_i n_j n_k n_l \ddot{I}_{kl} - 2n_j n_k \ddot{I}_{ik} - 2n_i n_k \ddot{I}_{jk}\right) \end{aligned} \tag{51}$$

(all retarded, as usual). This is the usual form for the amplitude of emitted gravitational waves.

It is straightforward to check that Eq. (51) corresponds to a transverse-traceless tensor. To see that it is transverse, note that

$$h_{ij}^{\rm GW} n_i = \frac{1}{R}\left(2\ddot{I}_{ij}n_i - \ddot{I}_{kk}n_j + n_i n_k n_l \ddot{I}_{kl} + n_j \ddot{I}_{kk} + n_j n_k n_l \ddot{I}_{kl} - 2n_j n_k n_i \ddot{I}_{ik} - 2n_k \ddot{I}_{jk}\right) = 0. \tag{52}$$

To see that it is traceless, take

$$h_{ii}^{\rm GW} = \frac{1}{R}\left(2\ddot{I}_{ii} - 3\ddot{I}_{kk} + 3n_k n_l \ddot{I}_{kl} + \ddot{I}_{kk} + n_k n_l \ddot{I}_{kl} - 2n_i n_k \ddot{I}_{ik} - 2n_i n_k \ddot{I}_{ik}\right) = 0. \tag{53}$$

C. The non-radiation terms

In addition to gravitational radiation, the metric can contain the non-radiation terms, the leading ones of which are associated with the mass M and angular momentum ${}^{(3)}\boldsymbol{S}$:

$$ds^2 = -\left(1 - 2\frac{M}{R}\right)dt^2 + 4\epsilon_{ijk}n_j\frac{S_k}{R^2}dt\,dx^i + \left(1 + 2\frac{M}{R}\right)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + ... \tag{54}$$

The "mass" terms M we have already discussed extensively: indeed, the mass M of a system is **measurable** using ordinary Newtonian dynamics, e.g. using Kepler's 3rd law.

The "angular momentum" terms ${}^{(3)}\boldsymbol{S}$ are a bit trickier: they lead to *gravitomagnetism*. They can be measured by their effect on moving objects, for example:

- The precession of the orbit of a satellite (as found on Homework #5).
- The precession of a gyroscope (the magnetic part of the Weyl tensor).

As an example of the latter, let's consider a point on the +3 axis ($x^1 = x^2 = 0$, $x^3 > 0$) and find the radial-radial component of the magnetic part of the Weyl tensor:

$$\mathcal{B}_{\hat{3}\hat{3}} = C_{\hat{3}\hat{0}\hat{1}\hat{2}} = C_{3012} = R_{3012} \tag{55}$$

to first order (the last equality holds in vacuum). Then (using Exercise 18.1 of MTW)

$$\mathcal{B}_{\hat{3}\hat{3}} = \frac{1}{2}(h_{32,10} + h_{10,32} - h_{02,31} - h_{31,02}) = \frac{1}{2}(h_{10,32} - h_{02,31}). \tag{56}$$

But on the 3-axis:

$$h_{10,32} = 2\left(S_3\frac{n_2}{R^2} - S_2\frac{n_3}{R^2}\right)_{,23} = 2S_3\left(\frac{n_2}{R^2}\right)_{,23} = 2S_3(R^{-3})_{,3} = -6\frac{S_3}{R^4}, \tag{57}$$

and similarly for the other term. We thus find

$$\mathcal{B}_{\hat{3}\hat{3}} = -6\frac{S_3}{R^4}. \tag{58}$$

Recall from HW#4 that the magnetic part of the Weyl tensor describes the relative precession of two sets of gyroscopes. Therefore, if we drop a spacecraft down toward an object, the gyroscopes in its nose and in its tail precess relative to each other at a rate given by the angular momentum of the object. The component of precession around the axis pointing in toward the object is determined by that component of the object's angular momentum!

This is the origin of *frame dragging* – the effect in which a massive rotating object causes inertial frames near it to precess relative to those farther away. It has many manifestations of which the gyroscope example above is the simplest.