Lecture IX: Field equations, cosmological constant, and tides

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I. OVERVIEW

We are now ready to construct Einstein's field equations, and examine the limit in which Newtonian gravity is recovered. The relation with Newtonian gravity enables us to determine the coefficient in the field equations. We will normally set $G = 1$ in this course.

The recommended reading for this lecture is:

- MTW Ch. 17.

II. FIELD EQUATIONS

In electromagnetism we found that the field strength tensor, the 2-form $\mathbf{F}$ satisfying the closure relation $F_{[\alpha\beta,\gamma]} = 0$, was related to the current density J^μ via the simple relation

$$F^{\mu\nu}{}_{,\nu} = 4\pi J^\mu. \tag{1}$$

We guessed the field equation from Gauss's law, but more generally it has the following key properties:

- It is *linear in the derivatives of the field strength*.
- It enforces *automatic conservation of the source*, $J^\mu{}_{,\mu} = 0$.
- The field strength is *zero in the absence of sources*.

We will now try to do the same for gravity.

A. Building the field equation

We wish to construct an equation

$$H^{\mu\nu} = T^{\mu\nu} \tag{2}$$

where $\mathbf{T}$ is the stress-energy tensor and $\mathbf{H}$ is some tensor derived from geometry. We wish to apply the same rules – i.e. we want $\mathbf{H}$ to be a symmetric 2nd rank tensor that is (i) linear in the highest derivative of the metric tensor that is used; (ii) automatically divergenceless, $H^{\mu\nu}{}_{;\nu} = 0$; and (iii) vanish when spacetime is flat (the solution with no sources). We have already constructed one such tensor: the Einstein tensor $G^{\mu\nu}$, which is linear in the second derivatives of the metric. Any multiple thereof will also do, so we write

$$G^{\mu\nu} = \kappa T^{\mu\nu}, \tag{3}$$

where κ is an undetermined constant. This (with $\kappa = 8\pi$) is the *Einstein field equation* (EFE).

The EFE is a system of 10 second-order PDEs for the 10 components of the metric. However, it is not as simple as the 10 components of $T^{\mu\nu}$ enabling us to solve the 10 components of $g_{\mu\nu}$, for three reasons.

- First, the stress-energy tensor describes matter, and matter moves according to the laws of physics that play out in curved spacetime: the source depends on the metric.

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- Since $G^{\mu\nu}{}_{;\nu} = 0$ for any metric, 4 components of the EFE are redundant: only 6 are functionally independent. This is unusual for an equation that describes 10 unknowns.
- Finally, there is an ambiguity in describing the metric: a coordinate system or *gauge* must be selected. There are 4 degrees of freedom in choosing the coordinates, since x^0, x^1, x^2, and x^3 are functionally independent. In order to solve the EFE a *gauge-fixing condition* must be introduced, which provides 4 new equations describing how the coordinates are to be chosen. This brings the total number of equations back to 10.

B. The value of the constant κ

How are we to determine the constant κ? It is really a fundamental constant of nature, but it is related to the familiar Newtonian gravitation constant G. In units where we fix the value of G (here we take $G = 1$), κ takes on a definite numerical value. We may determine it by reference to Newtonian physics, where the relative acceleration of nearby particles is related via the Riemann tensor to the Einstein tensor and hence the stress-energy tensor.

Consider an observer freely falling in a medium of some density ρ and negligible velocity ($v \ll 1$) and pressure ($p \ll \rho$). Then our observer sees in their local Lorentz frame the stress-energy tensor

$$\mathbf{T} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4}$$

It follows that the Einstein tensor must be

$$\mathbf{G} = \begin{pmatrix} \kappa\rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5}$$

We may determine the Ricci tensor that the observer sees by the following simple argument. Take the definition of the Einstein tensor,

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} \tag{6}$$

and lower an index:

$$G^{\mu}{}_{\nu} = R^{\mu}{}_{\nu} - \frac{1}{2} R \delta^{\mu}{}_{\nu}. \tag{7}$$

Taking the trace gives

$$G = R - \frac{1}{2} n R, \tag{8}$$

where $G \equiv G^{\alpha}{}_{\alpha}$ and $n = 4$ is the dimensionality of spacetime (the trace of the identity!). Then we find that

$$R = -\frac{2}{n-2} G, \tag{9}$$

and so

$$R^{\mu\nu} = G^{\mu\nu} - \frac{1}{n-2} G g^{\mu\nu}. \tag{10}$$

Using our expression above for the Einstein tensor, we have in the local Lorentz frame of the observer

$$G^{\hat{0}\hat{0}} = \kappa\rho, \quad \text{other components zero} \quad \rightarrow \quad G = -\kappa\rho \tag{11}$$

and then

$$R^{\hat{0}\hat{0}} = \left(1 - \frac{1}{n-2}\right) \kappa\rho = \frac{n-3}{n-2} \kappa\rho. \tag{12}$$

Now recall that our observer may use the Riemann tensor to measure the relative accelerations of test particles near their location whose infinitesimal displacement is $\boldsymbol{\xi}$:

$$\frac{D^2\xi^\alpha}{d\tau^2} = -R^\alpha{}_{\beta\gamma\delta} u^\beta \xi^\gamma u^\delta. \tag{13}$$

The spatial components of this as seen by our observer with 4-velocity $(1, 0, 0, 0)$ are

$$\frac{D^2\xi^{\hat{i}}}{d\tau^2} = -R^{\hat{i}}{}_{\hat{0}\hat{j}\hat{0}} \xi^{\hat{j}}. \tag{14}$$

An observer using Newtonian physics therefore sees a *gravity gradient*

$$\frac{\partial^{(n-1)} a^{\hat{i}}}{\partial x^{\hat{j}}} = -R^{\hat{i}}{}_{\hat{0}\hat{j}\hat{0}}, \tag{15}$$

where ${}^{(n-1)}\boldsymbol{a}$ is the "gravitational field" (a Newtonian concept). We then take the divergence of this field, and note that $R^{\hat{0}}{}_{\hat{0}\hat{0}\hat{0}} = 0$ to obtain

$${}^{(n-1)}\boldsymbol{\nabla} \cdot {}^{(n-1)}\boldsymbol{a} = -R^{\hat{i}}{}_{\hat{0}\hat{i}\hat{0}} = -R_{\hat{0}\hat{0}} = -R^{\hat{0}\hat{0}} = -\frac{n-3}{n-2}\kappa\rho. \tag{16}$$

Now we are in a position to ask what is the gravitational field that would be computed around a spherical object of radius r and uniform density ρ? It is:

$${}^{(n-1)}a = \frac{1}{n-1}({}^{(n-1)}\boldsymbol{\nabla} \cdot {}^{(n-1)}\boldsymbol{a})r = -\frac{n-3}{(n-1)(n-2)}\kappa\rho r, \tag{17}$$

where the $-$ sign tells us that the acceleration points inward for positive $\kappa\rho$. So for gravity to be attractive we want κ to be positive. But we can do better: we know that in $n-1$ spatial dimensions, a sphere has volume

$$\text{Volume} = v_{n-1} r^{n-1}, \tag{18}$$

and hence we may write the density in terms of the mass,

$$\rho = \frac{m}{v_{n-1} r^{n-1}}. \tag{19}$$

Thus

$${}^{(n-1)}a = -\frac{n-3}{(n-1)(n-2)v_{n-1}}\kappa\frac{m}{r^{n-2}}. \tag{20}$$

Note that, amazingly, the EFE has predicted the inverse-square law (for the empirically relevant case $n = 4$). Moreover, we may find κ by requiring the coefficient in Newton's law of gravitation to be unity:

$$\kappa = \frac{(n-1)(n-2)}{n-3} v_{n-1} = \frac{(3)(2)}{1}\frac{4}{3}\pi = 8\pi. \tag{21}$$

Thus in 4 dimensions:

$$G^{\mu\nu} = 8\pi T^{\mu\nu}. \tag{22}$$

Note further that lower numbers of dimensions are pathological: Eq. (20) tells us that a spherical mass in 3 total dimensions (2 spatial + 1 time) would not gravitate at all, regardless of κ. Worse, in 2 total dimensions (1 spatial + 1 time) there is a zero in the denominator: the massive sphere is not even allowed to exist! This can be traced to the remarkable fact that $G = 0$ for any 2-dimensional manifold. Fortunately, our universe is 3+1 dimensional so we need not worry about these pathologies.

III. COSMOLOGICAL CONSTANT

Einstein postulated that the Universe, despite being filled with matter, was static. This caused a serious problem for his theory. Let us return to Eq. (16), with $n = 4$ and $\kappa = 8\pi$:

$$^{(3)}\boldsymbol{\nabla} \cdot {}^{(3)}\boldsymbol{a} = -4\pi\rho. \tag{23}$$

By making the Universe static – the various galaxies appear not to accelerate relative to each other – Einstein forced the left hand side to be zero. The right hand side, however, is negative if the cosmos is filled with matter.

Einstein resolved the contradiction by modifying the theory. He had to give up one of our assumptions above about $H^{\mu\nu}$ in order to accomplish a modification; but which one? The entire structure of the theory would change if we sacrifice the assumption of linearity in the second derivatives, and certainly dropping conservation of energy-momentum would be even more radical. The minimalist modification to the EFE would be to drop the assumption that empty spacetime is flat. Then, since in general $g^{\mu\nu}{}_{;\nu} = 0$, one could write a field equation

$$G^{\mu\nu} + \Lambda g^{\mu\nu} = 8\pi T^{\mu\nu}, \tag{24}$$

where Λ is a small number called the *cosmological constant*. If Λ is constant then the source is still automatically conserved, and the left-hand side is a symmetric 2nd rank tensor.

The cosmological constant changes the analysis of the Newtonian limit of the theory. Now we have

$$\mathbf{G} = \begin{pmatrix} 8\pi\rho + \Lambda & 0 & 0 & 0 \\ 0 & -\Lambda & 0 & 0 \\ 0 & 0 & -\Lambda & 0 \\ 0 & 0 & 0 & -\Lambda \end{pmatrix}. \tag{25}$$

This time we have

$$G = -8\pi\rho - 4\Lambda, \tag{26}$$

and so

$$R^{\hat{0}\hat{0}} = 8\pi\rho + \Lambda + \frac{1}{2}(-8\pi\rho - 4\Lambda) = 4\pi\rho - \Lambda. \tag{27}$$

Thus the divergence of the apparent gravitational field that would be measured by an observer using Newtonian laboratory techniques is now:

$$^{(3)}\boldsymbol{\nabla} \cdot {}^{(3)}\boldsymbol{a} = -4\pi\rho + \Lambda. \tag{28}$$

You can now see that positive Λ is repulsive: it causes test particles to fly apart, just like negative matter density. So Einstein hypothesized that the Universe was balanced between the attractive pull of matter and the mysterious Λ term, whose value was 4π times the mean cosmic density.

We now know that the Universe is expanding and there is no such balance. For many years it was also believed that the expanding Universe did away with the need for the Λ term. However, observations of the expansion history of the Universe (which we will cover later) in fact show positive acceleration, which requires that the Λ term be re-introduced. So far as we can tell, cosmological observations are consistent with Λ as the single modification to gravity, with a value [1]

$$\Lambda = (1.27 \pm 0.07) \times 10^{-56}\,\text{cm}^{-2} = (1.14 \pm 0.06) \times 10^{-35}\,\text{s}^{-2} = (2.76 \pm 0.15) \times 10^{-46}\,M_\odot^{-2}. \tag{29}$$

This is such a small value that it has no significant effect on laboratory physics, or stellar astrophysics, or the observable properties of black holes. Only in the regions of lowest density, i.e. in the setting of cosmology, does Λ play a role.

IV. TIDAL FIELDS

[This is not actually covered in MTW Ch. 17, but it's an aspect of the structure of GR that we'll use again so I'll discuss it here.]

In our earlier analysis, we showed that the apparent gravity gradient $S^{\hat{i}}{}_{\hat{j}} = \partial a^{\hat{i}}/\partial x^{\hat{j}}$ seen by an observer was related to the Riemann tensor:

$$S^{\hat{i}}{}_{\hat{j}} = \frac{\partial a^{\hat{i}}}{\partial x^{\hat{j}}} = -R^{\hat{i}}{}_{\hat{0}\hat{j}\hat{0}} = -R_{\hat{i}\hat{0}\hat{j}\hat{0}}. \tag{30}$$

The gravity gradient has 9(=3 × 3) components, and we've seen that one of them is related to the local density in the Newtonian case. Indeed, in the most general case, with the help of the EFE, we see that

$${}^{(3)}\boldsymbol{\nabla}\cdot\boldsymbol{a} = S^{\hat{i}}{}_{\hat{i}} = -R_{\hat{0}\hat{0}} = -G_{\hat{0}\hat{0}} + \frac{1}{2}(G_{\hat{0}\hat{0}} - G_{\hat{i}\hat{i}}) = -4\pi(T_{\hat{0}\hat{0}} + T_{\hat{i}\hat{i}}). \tag{31}$$

Therefore, we see that the isotropic component of the gravity gradient always depends only on the local stress-energy tensor. This also tells us that for a perfect fluid, ${}^{(3)}\boldsymbol{\nabla}\cdot\boldsymbol{a} = -4\pi(\rho + 3P)$: that is, **both density and pressure exert attractive gravity**.

What of the other eight components of **S**? The symmetries of the Riemann tensor tell us that

$$S_{\hat{i}\hat{j}} - S_{\hat{j}\hat{i}} = 0, \tag{32}$$

i.e. the gravity gradient matrix is symmetric. So really **S** only has 6 independent components. We have learned that **the apparent acceleration due to gravity, ${}^{(3)}\boldsymbol{a}$, has zero curl**. This is familiar from Newtonian physics, but now we see it arising in general from GR – indeed, simply from geometry, since we did not use the EFE to prove symmetry of **S**!

The other 5 components of **S** – i.e. the symmetric trace-free components – are a different story. The many symmetries of the Riemann tensor do not allow us to describe them in terms of the local stress-energy tensor. They describe *tidal fields*: relative acceleration of infinitesimally separated test particles that are caused by sources elsewhere in the Universe. The best known example of a tidal field is the gradient in gravitational field near the Earth produced by the Moon, but there are many other astrophysically important examples.

How are we to describe tidal fields in a relativistically covariant way in GR? By redefining them as the part of the full Riemann curvature tensor that is not determined locally by $T^{\mu\nu}$, or equivalently by

$$R^{\mu\nu} = \frac{1}{8\pi}\left(T^{\mu\nu} - \frac{1}{2}T^{\alpha}{}_{\alpha}g^{\mu\nu}\right). \tag{33}$$

We define the *Weyl curvature tensor* to be

$$C^{\alpha\beta}{}_{\gamma\delta} \equiv R^{\alpha\beta}{}_{\gamma\delta} - 2\delta^{[\alpha}{}_{[\gamma}R^{\beta]}{}_{\delta]} + \frac{1}{3}\delta^{[\alpha}{}_{[\gamma}\delta^{\beta]}{}_{\delta]}R. \tag{34}$$

By construction, this has the same symmetries as the Riemann tensor (this follows from the symmetry of $\delta^{\alpha}{}_{\beta}$ and $R^{\alpha}{}_{\beta}$). Moreover, it is traceless in the sense that

$$\begin{aligned} C^{\alpha\beta}{}_{\alpha\delta} &= R^{\beta}{}_{\delta} - 2\delta^{[\alpha}{}_{[\alpha}R^{\beta]}{}_{\delta]} + \frac{1}{3}\delta^{[\alpha}{}_{[\alpha}\delta^{\beta]}{}_{\delta]}R \\ &= R^{\beta}{}_{\delta} - \frac{1}{2}\left(4R^{\beta}{}_{\delta} - R^{\beta}{}_{\delta} - R^{\beta}{}_{\delta} + \delta^{\beta}{}_{\delta}R\right) + \frac{1}{12}\left(4\delta^{\beta}{}_{\delta} - \delta^{\beta}{}_{\delta} - \delta^{\beta}{}_{\delta} + 4\delta^{\beta}{}_{\delta}\right)R \\ &= 0. \end{aligned} \tag{35}$$

This set of conditions implies that the Weyl tensor has 10 independent components (the 20 of the Riemann tensor, minus the 10 restrictions implied by tracelessness). The 10 components of the Weyl tensor and the 10 of the Ricci tensor are equivalent to specifying the Riemann tensor.

The gravity gradient seen by a given observer has components

$$S_{\hat{i}\hat{j}} = -R_{\hat{i}\hat{0}\hat{j}\hat{0}} = -C_{\hat{i}\hat{0}\hat{j}\hat{0}} - \frac{1}{2}\delta_{\hat{i}\hat{j}}R_{\hat{0}\hat{0}} - \frac{1}{2}R_{\hat{i}\hat{j}} + \frac{1}{6}\delta_{\hat{i}\hat{j}}R, \tag{36}$$

and its traceless part is

$$S_{\hat{i}\hat{j}} - \frac{1}{3}S_{\hat{k}\hat{k}}\delta_{\hat{i}\hat{j}} = -C_{\hat{i}\hat{0}\hat{j}\hat{0}} - \frac{1}{2}\left(R_{\hat{i}\hat{j}} - \frac{1}{3}R_{\hat{k}\hat{k}}\delta_{\hat{i}\hat{j}}\right). \tag{37}$$

Far away from any matter sources, the tidal field seen by an observer is a component of $C_{\alpha\beta\gamma\delta}$. Which component is relevent depends, of course, on the observer's 4-velocity (local Lorentz frame).

The Weyl tensor describes all relative accelerations produced remotely by distant matter. It is therefore useful not just for studying planetary tides, but also other situations such as gravitational waves.

The 5 independent components $C_{\hat{\imath}\hat{0}\hat{\jmath}\hat{0}}$ are measurable by their effect on test particles that are stationary in the observer's frame. For this reason, they are often said to form the *electric part* of the Weyl tensor:

$$\mathcal{E}_{\hat{\imath}\hat{\jmath}} = C_{\hat{\imath}\hat{0}\hat{\jmath}\hat{0}}. \tag{38}$$

This leaves 5 more components, which are said to form the *magnetic part* of the Weyl tensor:

$$\mathcal{B}_{\hat{\imath}\hat{\jmath}} = \frac{1}{2}\epsilon_{\hat{\jmath}\hat{k}\hat{l}}C_{\hat{\imath}\hat{0}\hat{k}\hat{l}}. \tag{39}$$

This is also traceless and symmetric (prove this!). The magnetic part of the Weyl tensor describes the gravitational torque on a gyroscope displaced infinitesimally from the observer's world line. As such it is the closest one comes to the gravitational analogue of "magnetic field gradient."

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[1] WMAP 7-year result, including BAO and H_0 determinations; $\Omega_\Lambda h^2 = .36 \pm 0.02$. Note that $\Lambda = 3\Omega_\Lambda H_0^2$.
`http://lambda.gsfc.nasa.gov/product/map/dr4/best_params.cfm`