Lecture VIII: More curvature tensors

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I. OVERVIEW

This lecture will complete our description of curvature tensors. In particular, we will build the Einstein tensor $G_{\mu\nu}$ that occupies the left-hand side of the field equations of general relativity. We will also prove two very important results:

- The Einstein equation is divergenceless, $G^{\mu\nu}{}_{;\nu} = 0$.
- Any spacetime with zero Riemann tensor is (at least locally) equivalent to flat spacetime.

II. BUILDING THE EINSTEIN TENSOR

In electrodynamics, we found that Maxwell's equation $F^{\mu\nu}{}_{,\nu} = 4\pi J^\mu$ implied the local conservation of charge: $J^\mu{}_{,\mu} = 0$. Equivalently, we formed a tensor $F^{\mu\nu}{}_{,\nu}$ out of the field strength tensor that (i) was of the appropriate type to equal a current – i.e. it had rank 1 – and (ii) was divergenceless regardless of the nature of the field (i.e. for any antisymmetric rank 2 tensor $F^{\mu\nu}$). We want to do the same for gravity. In the case of gravity, we have a "source" that is the stress-energy tensor, $T^{\mu\nu}$, that is divergenceless if energy and momentum are locally conserved:

$$T^{\mu\nu}{}_{;\nu} = 0. \tag{1}$$

It therefore seems natural for us to try to build a rank 2 symmetric tensor $G^{\mu\nu}$, constructed geometrically on a differentiable manifold with metric, that automatically satisfies

$$G^{\mu\nu}{}_{;\nu} = 0. \tag{2}$$

We will then be able to write a field equation $\mathbf{G} = \kappa\mathbf{T}$, where κ is some proportionality constant (the analogue of 4π in E&M).

We already know of one divergenceless rank 2 symmetric tensor: the metric tensor itself. However we can't possibly write an equation like $\mathbf{g} = \kappa\mathbf{T}$ since in empty flat spacetime $\mathbf{T} = 0$ but $\mathbf{g} \neq 0$. So we need to find a curvature tensor. The Riemann tensor is a place to start, but it has 4 indices, not 2. Thus we need to find a way to construct the tensor $\mathbf{G}$ that we seek algebraically out of the Riemann tensor. To do this, we need to consider the Riemann tensor's derivative properties.

A. First Bianchi identity

The covariant derivative of the Riemann tensor is the rank 5 tensor $R^\alpha{}_{\beta\gamma\delta;\epsilon}$. It is most convenient to prove theorems about this if we write this in a local Lorentz frame at point $\mathcal{P}$. Thus at $\mathcal{P}$ all Γ's are zero, so we may replace the semicolon by a comma:

$$R^\alpha{}_{\beta\gamma\delta;\epsilon}(\mathcal{P}) = (\Gamma^\alpha{}_{\beta\delta,\gamma} - \Gamma^\alpha{}_{\beta\gamma,\delta} + \Gamma^\alpha{}_{\nu\gamma}\Gamma^\nu{}_{\beta\delta} - \Gamma^\alpha{}_{\nu\delta}\Gamma^\nu{}_{\beta\gamma})_{,\epsilon}. \tag{3}$$

Since the Γ's are zero at $\mathcal{P}$, the derivatives of their products are zero (think of the product rule), so in the local Lorentz frame:

$$R^\alpha{}_{\beta\gamma\delta;\epsilon}(\mathcal{P}) = \Gamma^\alpha{}_{\beta\delta,\gamma\epsilon} - \Gamma^\alpha{}_{\beta\gamma,\delta\epsilon}. \tag{4}$$

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It now follows that if we completely antisymmetrize in the last 3 indices $[\gamma\delta\epsilon]$, then the partial derivatives on the right-hand side become zero (remember that partial derivatives commute):

$$R^\alpha{}_{\beta[\gamma\delta;\epsilon]}(\mathcal{P}) = 0. \tag{5}$$

Since we now have a tensor on both sides this is valid in any coordinate system. Therefore the particular combination of the Riemann tensor derivatives is zero.

B. Ricci tensor

Now that we know something about the derivatives of the Riemann tensor it's time to start building rank 2 versions thereof. The easiest way to build a rank 2 tensor from the Riemann tensor is via contraction. This gives the *Ricci tensor*:

$$R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}. \tag{6}$$

(Other possible contractions are equivalent or are automatically zero by symmetry, e.g. $R^\alpha{}_{\alpha\mu\nu} = 0$.) This is a symmetric rank 2 tensor. It even has its own contraction, known as the *Ricci scalar*:

$$R = R^\beta{}_\beta = R^{\alpha\beta}{}_{\alpha\beta}. \tag{7}$$

Is the Ricci tensor the divergenceless object we seek? Unforunately no, as we can see by contracting the first Bianchi identity. Using the antisymmetry of the Riemann tensor on the last two indices, we can write Eq. (5) as

$$R^\alpha{}_{\beta\gamma\delta;\epsilon} + R^\alpha{}_{\beta\delta\epsilon;\gamma} + R^\alpha{}_{\beta\epsilon\gamma;\delta} = 0, \tag{8}$$

or if we contract on the 1st and 3rd indices:

$$R_{\beta\delta;\epsilon} + R^\alpha{}_{\beta\delta\epsilon;\alpha} + R^\alpha{}_{\beta\epsilon\alpha;\delta} = 0. \tag{9}$$

Using the antisymmetry of the Riemann tensor we can manipulate the last term into

$$R_{\beta\delta;\epsilon} + R^\alpha{}_{\beta\delta\epsilon;\alpha} - R_{\beta\epsilon;\delta} = 0. \tag{10}$$

If we now contract on β and ϵ, we find

$$R^\beta{}_{\delta;\beta} + R^{\alpha\beta}{}_{\delta\beta;\alpha} - R_{;\delta} = 0. \tag{11}$$

The symmetries of the Riemann tensor tell us that $R^{\alpha\beta}{}_{\delta\beta} = R^{\beta\alpha}{}_{\beta\delta} = R^\alpha{}_\delta$, so

$$R^\beta{}_{\delta;\beta} + R^\alpha{}_{\delta;\alpha} - R_{;\delta} = 0 \tag{12}$$

or

$$R^\alpha{}_{\delta;\alpha} = \frac{1}{2} R_{;\delta}. \tag{13}$$

So it turns out that the Ricci tensor is in general not divergenceless. Therefore we do not want to set it equal to $\mathbf{T}$.

C. Einstein tensor

This deficiency of the Ricci tensor as an object to put on the left-hand side of the field equation is however easily repaired. Recall that the metric tensor has zero covariant derivative. Therefore if we define the rank 2 symmetric *Einstein tensor*

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}, \tag{14}$$

we see that

$$G^\alpha{}_{\delta;\alpha} = R^\alpha{}_{\delta;\alpha} - \frac{1}{2} R_{;\alpha} g^\alpha{}_\delta = \frac{1}{2} R_{;\delta} - \frac{1}{2} R_{;\delta} = 0. \tag{15}$$

We thus learn that **the Einstein tensor has zero divergence**. It is therefore appropriate for us to expect a relation such as

$$G^{\mu\nu} = \kappa T^{\mu\nu} \tag{16}$$

for some constant κ (which will turn out to be 8π in units where Newton's gravitaitonal constant is unity).

We will look at the structure of GR much more in the future, but you can see already that $G^{\mu\nu}$ has 10 components, which are linear combinations of the 20 components of $R^{\alpha\beta\gamma\delta}$. These 10 components of the curvature of spacetime are determined locally by the matter content. The other 10 components are determined nonlocally, through the fact that a single metric tensor $g_{\mu\nu}$ generates all of the Riemann tensor.

III. SPACETIME WITH ZERO RIEMANN TENSOR IS FLAT

Thus far we have referred to the Riemann tensor as the "curvature" of spacetime. But what we haven't done yet is prove that the Riemann tensor contains all of the information about curvature. More precisely: suppose we consider a spacetime $\mathcal{M}$ with some messy metric $g_{\mu\nu}(x^\alpha)$, but with zero Riemann tensor. Is it necessarily true that it is flat spacetime? In this section we will prove that the answer is **yes**. We will do so by first constructing a special type of coordinate system – *Riemann normal coordinates* – and then showing that for zero Riemann tensor this coordinate system is that of ordinary Minkowski spacetime.

A. Riemann normal coordinates

[Note: You can read at this point MTW §11.6. While formally "Track 2," we have covered enough background for it.]

Let us consider a point $\mathcal{P}$ on a manifold, and suppose that we have constructed locally Lorentz coordinates at that point, i.e.

$$x^\alpha(\mathcal{P}) = 0, \quad g_{\alpha\beta}(0) = \eta_{\alpha\beta}, \quad \text{and} \quad g_{\alpha\beta,\gamma}(0) = 0. \tag{17}$$

This is only a local Lorentz frame, and we have said nothing yet about the higher partial derivatives of $g_{\alpha\beta}$ (or equivalently about the higher order terms in the Taylor expansion of x). Indeed these are not uniquely specified. We can however build a special primed coordinate system as follows. Given any vector $\boldsymbol{v}$ based at $\mathcal{P}$, let's launch a geodesic $\mathcal{G}(\lambda;\boldsymbol{v})$ from $\mathcal{P}$ with

$$\mathcal{G}(0;\boldsymbol{v}) = \mathcal{P} \quad \text{and} \quad \left.\frac{d}{d\lambda}\mathcal{G}(\lambda;\boldsymbol{v})\right|_{\lambda=0} = \boldsymbol{v}, \tag{18}$$

or in coordinate language

$$x^\mu(0;\boldsymbol{v}) = 0 \;\text{ and }\; \frac{dx^\mu}{d\lambda}(\lambda=0;\boldsymbol{v}) = v^\mu. \tag{19}$$

The geodesic can be evolved to any other value of λ by using the 2nd order ODE for $x^\mu(\lambda)$. In particular, we can evolve to $\lambda = 1$, and thus find a mapping from vectors at $\mathcal{P}$ to positions in the manifold $\mathcal{M}$:

$$\boldsymbol{v} \longmapsto \mathcal{G}(1;\boldsymbol{v}). \tag{20}$$

In words, this says: "Start at position $\mathcal{P}$ and with velocity $\boldsymbol{v}$; move along with zero acceleration for unit time; and then where you end up is $\mathcal{G}(1;\boldsymbol{v})$." As long as one is in a small portion of the manifold such that this mapping remains one-to-one, we may define a coordinate system by assigning the components v^μ as the new (primed) coordinates of the point $\mathcal{G}(1;v^\mu)$:

$$x^{\alpha'}[\mathcal{G}(1;\boldsymbol{v})] = \delta^{\alpha'}_{\mu} v^\mu. \tag{21}$$

Such a coordinate system is called a *Riemann normal coordinate* system.

We may study the general properties of Riemann normal coordinates. In particular, we may do a Taylor expansion of the geodesics:

$$x^\mu[\mathcal{G}(0;\boldsymbol{v})] = 0, \quad \frac{dx^\mu}{d\lambda}[\mathcal{G}(0;\boldsymbol{v})] = v^\mu, \quad \text{and} \quad \frac{d^2x^\mu}{d\lambda^2}[\mathcal{G}(0;\boldsymbol{v})] = 0. \tag{22}$$

Therefore, doing a Taylor expansion in λ (or equivalently in $\boldsymbol{v}$):

$$x^\mu(x^{\alpha'}) = v^\mu + \mathcal{O}(x'^3) = \delta^\mu_{\alpha'} x^{\alpha'} + \mathcal{O}(x'^3). \tag{23}$$

By converting the metric tensor to the primed coordinates, we find that

$$g_{\alpha'\beta'}(0) = \eta_{\alpha'\beta'}, \quad \text{and} \quad g_{\alpha'\beta',\gamma'}(0) = 0. \tag{24}$$

B. The metric in Riemann normal coordinates for the case where the Riemann tensor is zero

What are the general properties of the metric in Riemann normal coordinates? To answer this question, let's take any point and construct the coordinate basis vector. The $\boldsymbol{e}_{\alpha'}$ basis vector is defined as the derivative of position with respect to the $x^{\alpha'}$ coordinate, or

$$\boldsymbol{e}_{\alpha'} = \delta^\mu_{\alpha'} \frac{\partial}{\partial v^\mu} \mathcal{G}(1, v^\mu). \tag{25}$$

Then a general vector $\boldsymbol{w}$ based at $\mathcal{G}(1, v^\mu)$ is expressed in terms of its components $\boldsymbol{w} = w^{\alpha'} \boldsymbol{e}_{\alpha'}$, or

$$\boldsymbol{w} = w^{\alpha'} \delta^\mu_{\alpha'} \frac{\partial}{\partial v^\mu} \mathcal{G}(1, v^\mu). \tag{26}$$

Note that $\boldsymbol{w}$ is expressed in terms of the infinitesimal displacement of geodesics. If we write the vector field $\boldsymbol{\xi}(\lambda)$ as the infinitesimal separation vector of two geodesics originating from $\mathcal{P}$ but with initial velocities v^μ and

$$v^\mu + \delta v^\mu = v^\mu + w^{\alpha'} \delta^\mu_{\alpha'}, \tag{27}$$

then we have $\boldsymbol{\xi}(1) = \boldsymbol{w}$.

If the Riemann tensor vanishes, then the geodesic equation gives us

$$\frac{D^2 \boldsymbol{\xi}(\lambda)}{d\lambda^2} = 0. \tag{28}$$

We then find that

$$\frac{d^3}{d\lambda^3}(\boldsymbol{\xi} \cdot \boldsymbol{\xi}) = 2\boldsymbol{\xi} \cdot \frac{D^3 \boldsymbol{\xi}}{d\lambda^3} + 2\frac{D\boldsymbol{\xi}}{d\lambda} \cdot \frac{D^2 \boldsymbol{\xi}}{d\lambda^2} = 0, \tag{29}$$

so $\boldsymbol{\xi} \cdot \boldsymbol{\xi}$ is a 2nd order polynomial in λ:

$$\boldsymbol{\xi} \cdot \boldsymbol{\xi} = a + b\lambda + c\lambda^2. \tag{30}$$

However, we may compute these coefficients via a Taylor expansion. We know that

$$\xi^\mu = w^{\alpha'} \delta^\mu_{\alpha'} \lambda + \mathcal{O}(\lambda^2), \tag{31}$$

and so

$$\boldsymbol{\xi} \cdot \boldsymbol{\xi} = g_{\mu\nu} \xi^{\mu\nu} = \eta_{\mu\nu} w^{\alpha'} \delta^\mu_{\alpha'} w^{\beta'} \delta^\nu_{\beta'} \lambda^2 + \mathcal{O}(\lambda^3). \tag{32}$$

The λ^3 and higher terms must vanish for a quadratic polynomial, so we conclude that

$$\boldsymbol{\xi} \cdot \boldsymbol{\xi} = \eta_{\alpha'\beta'} w^{\alpha'} w^{\beta'} \lambda^2 \tag{33}$$

and hence

$$\boldsymbol{w} \cdot \boldsymbol{w} = \eta_{\alpha'\beta'} w^{\alpha'} w^{\beta'}. \tag{34}$$

Thus we conclude that the metric tensor in the primed coordinates is simply $\eta_{\alpha'\beta'}$ and hence the Riemann normal coordinate system is that of special relativity.

[**Exception**: We have only shown that $R_{\alpha\beta\gamma\delta} = 0$ implies that spacetime is flat in some finite region around any point. In particular, we have not proven that the global structure or topology of spacetime is that of $\mathbb{M}^4$ – indeed, it may not be.]