Lecture VII: Geodesics and curvature

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I. OVERVIEW

In this lecture, we will investigate geodesics. These can be defined in two ways – as curves of zero acceleration, and as curves of stationary length. We will show these to be equivalent. We will then define the Riemann curvature tensor and relate it to the behavior of neighboring geodesics.

The recommended reading for this lecture is:

- MTW §8.7.

II. GEODESICS

Geodesics play a central role in GR: they are the trajectories followed by freely falling particles. So today we are finally ready to define and use them. We will define them first as curves of zero acceleration. We will follow this by showing that the same curves can be derived from an action principle: they are curves of stationary length.

A. Geodesics as zero acceleration curves

In flat spacetime, we defined the acceleration as $\boldsymbol{a} = d\boldsymbol{v}/d\tau$. The existence of the covariant derivative has made it possible to define the derivative of the velocity in curved spacetime. To do this in the most general way possible, consider any trajectory $\mathcal{P}(\lambda)$, parameterized by λ (which may be the proper time τ, but we won't impose this since we want to be able to treat photons as well), and with a tangent vector $\boldsymbol{v} = d\mathcal{P}/d\lambda$. If $\boldsymbol{S}$ is a general tensor field, we define its derivative along the curve as

$$\frac{D\mathbf{S}}{d\lambda} = \nabla_{d\mathcal{P}/d\lambda}\mathbf{S} = \nabla_{\boldsymbol{v}}\mathbf{S}, \tag{1}$$

or in component language for a $\binom{1}{1}$ tensor,

$$\frac{DS^\alpha{}_\beta}{d\lambda} = v^\gamma S^\alpha{}_{\beta;\gamma}. \tag{2}$$

The capital D is used here to remind us that we are using the covariant derivative (the lower case d would denote the derivative of the components $S^\alpha{}_\beta$ – usually not what we want). Expansion of the covariant derivative in terms of partial derivatives tells us that

$$\frac{DS^\alpha{}_\beta}{d\lambda} = v^\gamma S^\alpha{}_{\beta,\gamma} + v^\gamma \Gamma^\alpha{}_{\mu\gamma} S^\mu{}_\beta - v^\gamma \Gamma^\mu{}_{\beta\gamma} S^\alpha{}_\mu = \frac{dS^\alpha{}_\beta}{d\lambda} + v^\gamma \Gamma^\alpha{}_{\mu\gamma} S^\mu{}_\beta - v^\gamma \Gamma^\mu{}_{\beta\gamma} S^\alpha{}_\mu. \tag{3}$$

In this form, one can see that $D\mathbf{S}/d\lambda$ depends only on the value of $\mathbf{S}$ along the trajectory. So it is perfectly valid to speak of the derivative of the tangent vector, $D\boldsymbol{v}/d\lambda$. In the special case of a curve parameterized by proper time, $\boldsymbol{v} = d\mathcal{P}/d\tau$ is the 4-velocity and $\boldsymbol{a} = D\boldsymbol{v}/d\tau$ is the 4-acceleration. In this case we have

$$\boldsymbol{a}\cdot\boldsymbol{v} = \frac{D\boldsymbol{v}}{d\tau}\cdot\boldsymbol{v} = \frac{1}{2}\frac{d}{d\tau}(\boldsymbol{v}\cdot\boldsymbol{v}) = 0. \tag{4}$$

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We now consider curves of zero acceleration, i.e. for which $D\boldsymbol{v}/d\lambda = 0$. Such a curve will be called a *geodesic*. Writing $v^\alpha = dx^\alpha/d\lambda$, we see that

$$\frac{Dv^\alpha}{d\lambda} = \frac{dv^\alpha}{d\lambda} + v^\gamma \Gamma^\alpha{}_{\mu\gamma} v^\mu = \frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha{}_{\mu\gamma}\frac{dx^\mu}{d\lambda}\frac{dx^\gamma}{d\lambda}. \tag{5}$$

If we set this equal to zero, we get a 2nd order system of ODEs for $x^\alpha(\lambda)$:

$$\frac{d^2x^\alpha}{d\lambda^2} = -\Gamma^\alpha{}_{\mu\gamma}\frac{dx^\mu}{d\lambda}\frac{dx^\gamma}{d\lambda}. \tag{6}$$

Note that in flat space ($\mathbb{R}^n$) or spacetime ($\mathbb{M}^4$) we find that in the standard coordinate systems all Γs are zero and x^α are linear functions of λ.

General definition: A tensor $\mathbf{S}$ is said to be *parallel-transported* along a curve if $D\mathbf{S}/d\lambda = 0$. As an example, the velocity for a geodesic is parallel-transported.

B. Geodesics as curves of stationary proper time

There is an alternative viewpoint of a geodesic which leads to the same conclusion. Suppose we have take a timelike curve $\mathcal{P}(\lambda)$. Let us suppose that the curve runs from $\mathcal{A} = \mathcal{P}(\lambda_\mathrm{i})$ to $\mathcal{B} = \mathcal{P}(\lambda_\mathrm{f})$. We want to find the conditions under which the proper time measured by an observer who follows the curve is stationary. To do this, write the total proper time along a trajectory

$$T = \int_{\lambda_\mathrm{i}}^{\lambda_\mathrm{f}} \frac{d\tau}{d\lambda}\, d\lambda. \tag{7}$$

We want to determine the variation δT of T with respect to perturbations of the trajectory, $\delta x^\mu(\lambda)$. We will assume without loss of generality that the original curve is parameterized by proper time, $\lambda = \tau$ so that $|\boldsymbol{v}|^2 = -1$, but we cannot assume this for the perturbed curve (since there is no guarantee that the perturbed curve will have proper time interval $\lambda_\mathrm{f} - \lambda_\mathrm{i}$).

We can see first that

$$\frac{d\tau}{d\lambda} = \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}, \tag{8}$$

so the variation is

$$\delta T = \int_{\lambda_\mathrm{i}}^{\lambda_\mathrm{f}} \frac{\delta[-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}]}{2\sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}}\, d\lambda. \tag{9}$$

If we perturb around a curve with $d\tau/d\lambda = 1$, then the square root in the denominator is unity, and so

$$\begin{aligned}\delta T &= \int_{\lambda_\mathrm{i}}^{\lambda_\mathrm{f}} \left[-\frac{1}{2}g_{\mu\nu,\sigma}\delta x^\sigma \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} - \frac{1}{2}g_{\mu\nu}\frac{d\delta x^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} - \frac{1}{2}g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{d\delta x^\nu}{d\lambda}\right] d\lambda \\ &= \int_{\lambda_\mathrm{i}}^{\lambda_\mathrm{f}} \left[-\frac{1}{2}g_{\mu\nu,\sigma}\delta x^\sigma \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} - g_{\mu\nu}\frac{d\delta x^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right] d\lambda. \end{aligned} \tag{10}$$

Now if we fix the endpoints $\mathcal{A}$ and $\mathcal{B}$, then $\delta x^\mu = 0$ at λ_i and λ_f. Therefore we may use integration by parts on the last term,

$$\delta T = \int_{\lambda_\mathrm{i}}^{\lambda_\mathrm{f}} \left[-\frac{1}{2}g_{\mu\nu,\sigma}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} + \frac{d}{d\lambda}\left(g_{\sigma\nu}\frac{dx^\nu}{d\lambda}\right)\right]\delta x^\sigma\, d\lambda. \tag{11}$$

In order for the curve to have stationary proper time, we must have the quantity in [] always equal to zero. This gives us a differential equation,

$$-\frac{1}{2}g_{\mu\nu,\sigma}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} + \frac{d}{d\lambda}\left(g_{\sigma\nu}\frac{dx^\nu}{d\lambda}\right) = 0, \tag{12}$$

or

$$-\frac{1}{2}g_{\mu\nu,\sigma}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} + g_{\sigma\nu,\mu}\frac{dx^\nu}{d\lambda}\frac{dx^\mu}{d\lambda} + g_{\sigma\nu}\frac{d^2x^\nu}{d\lambda^2} = 0. \tag{13}$$

We now raise indices (multiply through by $g^{\kappa\sigma}$ – since this is invertible the original and final equations have equivalent information) and find

$$\frac{d^2x^\kappa}{d\lambda^2} = \frac{1}{2}g^{\kappa\sigma}\Big(-g_{\mu\nu,\sigma}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} + 2g_{\sigma\nu,\mu}\frac{dx^\nu}{d\lambda}\frac{dx^\mu}{d\lambda}\Big). \tag{14}$$

Using the Christoffel symbols, we can turn this into

$$\frac{d^2x^\kappa}{d\lambda^2} = -\Gamma^\kappa{}_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}. \tag{15}$$

This is equivalent to Eq. (6) so we conclude that the timelike curves of extremal proper time **are** the timelike geodesics!

[**Note**: We have shown that the timelike curves are stationary, but have not said whether they are maxima, minima, or saddle points. In Minkowski space it is easy to show that they are maxima. Saddle points are possible in more general spacetimes (not even exotic ones – the Earth's trajectory from its location in April 2010 to October 2011 was a saddle point). There are however no minima, since one can always add little wiggles to a trajectory to shorten its proper time.]

C. Example: Polar coordinates

In polar coordinates in $\mathbb{R}^2$, we showed that the nonzero Christoffel symbols were

$$\Gamma^r{}_{\theta\theta} = -r, \quad \text{and} \quad \Gamma^\theta{}_{\theta r} = \Gamma^\theta{}_{r\theta} = \frac{1}{r}. \tag{16}$$

It follows that the equation of motion for a geodesic is

$$\frac{d^2r}{d\lambda^2} = r\left(\frac{d\theta}{d\lambda}\right)^2 \quad \text{and} \quad \frac{d^2\theta}{d\lambda^2} = -\frac{2}{r}\frac{dr}{d\lambda}\frac{d\theta}{d\lambda}. \tag{17}$$

This equation is the first example we have seen of *inertial forces*: the apparent bending of a trajectory because of the coordinate system. In particular, you can see that $d^2r/d\lambda^2 \geq 0$: particles with any motion in the θ-direction appear to be repelled from the origin ("centrifugal force"). GR describes gravity entirely in such terms.

III. RIEMANN CURVATURE TENSOR

Thus far we have described how to compute geodesics (and, by implication, the trajectories of freely falling particles in any spacetime of known metric tensor). But we also said that in GR, matter causes spacetime "curvature." We have a way to describe how much matter there is – this is the stress-energy tensor $T_{\mu\nu}$ – but we don't yet have a way of quantifying curvature. This is our job now. We will approach the problem of curvature in two ways: one by considering the (non)commutativity of the covariant derivative, and one by considering the behavior of neighboring geodesics. The former lends itself to mathematical analysis, but the latter is most closely associated with our intuitive notion of how the curvature of spacetime is measured.

A. Switching the order of covariant derivatives

In ordinary calculus, you learned about the commutativity of partial derivatives for any smooth function,

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 f}{\partial y\,\partial x}. \tag{18}$$

In special relativity, this carried over to the commutativity of partial derivatives of an arbitrary tensor,

$$S^\alpha{}_{\beta,\gamma\delta} = S^\alpha{}_{\beta,\delta\gamma}. \tag{19}$$

An obvious question now is the behavior of covariant derivatives – in this case does the order matter? For scalars, the answer is no:

$$f_{;\beta\alpha} \equiv \nabla_\alpha \nabla_\beta f = \nabla_\alpha f_{,\beta} = f_{,\beta\alpha} - \Gamma^\mu{}_{\beta\alpha} f_{,\mu}, \tag{20}$$

and on account of the symmetry of the Christoffel symbol this is symmetric under interchange of α and β.

But for vectors, the story is different: if we have a vector field w^γ, then

$$\begin{aligned} w^\gamma{}_{;\beta\alpha} &\equiv \nabla_\alpha \nabla_\beta w^\gamma \\ &= \nabla_\alpha (w^\gamma{}_{,\beta} + \Gamma^\gamma{}_{\mu\beta} w^\mu) \\ &= (w^\gamma{}_{,\beta} + \Gamma^\gamma{}_{\mu\beta} w^\mu)_{,\alpha} + \Gamma^\gamma{}_{\nu\alpha}(w^\nu{}_{,\beta} + \Gamma^\nu{}_{\mu\beta} w^\mu) - \Gamma^\pi{}_{\beta\alpha}(w^\gamma{}_{,\pi} + \Gamma^\gamma{}_{\mu\pi} w^\mu) \\ &= w^\gamma{}_{,\beta\alpha} + \Gamma^\gamma{}_{\mu\beta,\alpha} w^\mu + \Gamma^\gamma{}_{\mu\beta} w^\mu{}_{,\alpha} + \Gamma^\gamma{}_{\nu\alpha} w^\nu{}_{,\beta} + \Gamma^\gamma{}_{\nu\alpha}\Gamma^\nu{}_{\mu\beta} w^\mu - \Gamma^\pi{}_{\beta\alpha} w^\gamma{}_{,\pi} - \Gamma^\pi{}_{\beta\alpha}\Gamma^\gamma{}_{\mu\pi} w^\mu. \end{aligned} \tag{21}$$

There are 7 terms here. Of these, terms #1, #6, and #7 are symmetric in α and β, and swapping α and β swaps terms #3 and #4. However, terms #2 and #5 are not symmetric. We therefore have

$$w^\gamma{}_{;\beta\alpha} - w^\gamma{}_{;\alpha\beta} = (\Gamma^\gamma{}_{\mu\beta,\alpha} + \Gamma^\gamma{}_{\nu\alpha}\Gamma^\nu{}_{\mu\beta} - \Gamma^\gamma{}_{\mu\alpha,\beta} - \Gamma^\gamma{}_{\nu\beta}\Gamma^\nu{}_{\mu\alpha}) w^\mu \tag{22}$$

Thus we see that in general covariant derivatives of vectors need not commute. The amount by which they do not commute depends only on the value of the vector field at the point in question (and not on its derivatives). The object in parentheses is called the *Riemann curvature tensor* (or "***Riemann***"). It is a tensor because the covariant derivatives were defined in such a way as to transform appropriately (i.e. according to the Jacobian) under changes of coordinate system.

So the Riemann tensor is a rank $\binom{1}{3}$ tensor, whose components are given by

$$R^\gamma{}_{\mu\alpha\beta} = \Gamma^\gamma{}_{\mu\beta,\alpha} + \Gamma^\gamma{}_{\nu\alpha}\Gamma^\nu{}_{\mu\beta} - \Gamma^\gamma{}_{\mu\alpha,\beta} - \Gamma^\gamma{}_{\nu\beta}\Gamma^\nu{}_{\mu\alpha}. \tag{23}$$

Thus

$$w^\gamma{}_{;\beta\alpha} - w^\gamma{}_{;\alpha\beta} = R^\gamma{}_{\mu\alpha\beta} w^\mu. \tag{24}$$

Since the Christoffel symbols depend on the metric and its 1st derivative, the Riemann tensor depends on the metric and its 2nd derivative. It has n^4 components (256 in 4-dimensional spacetime!) but most are zero or related by symmetries. For example, it is obvious that the Riemann tensor is antisymmetric on the last two indices,

$$R^\gamma{}_{\mu\alpha\beta} = -R^\gamma{}_{\mu\beta\alpha}. \tag{25}$$

The non-commutation of covariant derivatives is a phenomenon that does not occur for flat spacetime. It does not even occur for flat spacetime with curved coordinates, since the Riemann tensor is a tensor and hence could be computed to be 0 in Cartesian coordinates and then converted to any other system. Therefore it is appropriate to use ***Riemann*** as a measure of spacetime curvature.

B. Geodesic deviation

We are now in a position to ask what happens to neighboring geodesics. If we take two freely falling test particles, place them next to each other, and let them go, do they fall together? Do they fly apart? This question is intimately related to the subject of tidal fields, familiar from Newtonian gravity.

To begin, let us consider a geodesic $\mathcal{P}(\lambda)$, and consider a neighboring geodesic separated from it by an infinitesimal displacement vector $\boldsymbol{\xi}(\lambda)$. We want to know how $\boldsymbol{\xi}$ behaves as we move along the geodesic. To make the notion of "infinitesimal displacement" more precise, we consider a family of geodesics parameterized by some parameter $n \in \mathbb{R}$, $\mathcal{P}(\lambda, n)$, such that our fiducial geodesic is $\mathcal{P}(\lambda, 0)$ and the infinitesimal displacement is

$$\boldsymbol{\xi} = \frac{\partial \mathcal{P}(\lambda, n)}{\partial n}. \tag{26}$$

The fact that these curves are geodesics is captured by taking their 4-velocity $\boldsymbol{v} = \partial\mathcal{P}(\lambda, n)/\partial\lambda$ and writing

$$\frac{D\boldsymbol{v}}{\partial\lambda} = \nabla_{\boldsymbol{v}} \boldsymbol{v} = 0. \tag{27}$$

We will need to prove one more fact about the vector fields $\boldsymbol{v}$ and $\boldsymbol{\xi}$. If we take the directional derivative of one of them with respect to the other,

$$\nabla_{\boldsymbol{v}}\xi^\mu = \nabla_{\boldsymbol{v}}\frac{\partial x^\mu}{\partial n} = \frac{\partial}{\partial\lambda}\frac{\partial x^\mu}{\partial n} + \Gamma^\mu{}_{\alpha\beta}\frac{\partial x^\alpha}{\partial\lambda}\frac{\partial x^\beta}{\partial n}. \tag{28}$$

In comparison,

$$\nabla_{\boldsymbol{\xi}}v^\mu = \nabla_{\boldsymbol{\xi}}\frac{\partial x^\mu}{\partial\lambda} = \frac{\partial}{\partial n}\frac{\partial x^\mu}{\partial\lambda} + \Gamma^\mu{}_{\alpha\beta}\frac{\partial x^\alpha}{\partial n}\frac{\partial x^\beta}{\partial\lambda}. \tag{29}$$

Symmetry of the partial derivative and the Christoffel symbol then tells us that

$$\nabla_{\boldsymbol{v}}\boldsymbol{\xi} = \nabla_{\boldsymbol{\xi}}\boldsymbol{v}. \tag{30}$$

(Two vector fields with this property are said to *commute*.)

Our ambition is to learn something about how $\boldsymbol{\xi}$ varies with λ. A natural thing to do is to take its derivative $D\boldsymbol{\xi}/\partial\lambda$. But if Newtonian gravity is a guide, the motion of infinitesimally separated test particles should be described by a 2nd order ODE, so let's take another derivative:

$$\begin{aligned}
\frac{D^2\xi^\gamma}{\partial\lambda^2} &= \nabla_{\boldsymbol{v}}\nabla_{\boldsymbol{v}}\xi^\gamma \\
&= \nabla_{\boldsymbol{v}}\nabla_{\boldsymbol{\xi}}v^\gamma \\
&= v^\alpha\nabla_\alpha(\xi^\beta\nabla_\beta v^\gamma) \\
&= (v^\alpha\nabla_\alpha\xi^\beta)\nabla_\beta v^\gamma + v^\alpha\xi^\beta\nabla_\alpha\nabla_\beta v^\gamma \\
&= (\xi^\alpha\nabla_\alpha v^\beta)\nabla_\beta v^\gamma + v^\alpha\xi^\beta(\nabla_\beta\nabla_\alpha v^\gamma + R^\gamma{}_{\mu\alpha\beta}v^\mu) \\
&= (\xi^\beta\nabla_\beta v^\alpha)\nabla_\alpha v^\gamma + v^\alpha\xi^\beta\nabla_\beta\nabla_\alpha v^\gamma + R^\gamma{}_{\mu\alpha\beta}v^\alpha\xi^\beta v^\mu \\
&= \xi^\beta\nabla_\beta(v^\alpha\nabla_\alpha v^\gamma) + R^\gamma{}_{\mu\alpha\beta}v^\alpha\xi^\beta v^\mu \\
&= R^\gamma{}_{\mu\alpha\beta}v^\alpha\xi^\beta v^\mu.
\end{aligned} \tag{31}$$

The upshot is that two nearby test particles appear to move relative to each other according to the Riemann tensor. Therefore it is really a description of tidal fields.

C. Properties of the Riemann tensor

Equation (23) gives us a way to compute all 256 components of the Riemann tensor. Unfortunately, this is a rather tedious process. The good news is that ***Riemann*** contains many symmetries that reduce the computational burden. One of these – the antisymmetry on the last two indices, Eq. (25):

$$R^\gamma{}_{\mu\alpha\beta} = -R^\gamma{}_{\mu\beta\alpha} \quad \text{or} \quad R^\gamma{}_{\mu(\alpha\beta)} = 0 \tag{32}$$

has already been identified, and reduces the number of components to $\frac{1}{2}n^3(n-1) = 96$.

A second property of the Riemann tensor that follows directly from Eq. (23) is that the antisymmetrization on the last 3 indices vanishes:

$$R^\gamma{}_{[\mu\alpha\beta]} = \Gamma^\gamma{}_{[\mu\beta,\alpha]} + \Gamma^\gamma{}_{\nu[\alpha}\Gamma^\nu{}_{\mu\beta]} - \Gamma^\gamma{}_{[\mu\alpha,\beta]} - \Gamma^\gamma{}_{\nu[\beta}\Gamma^\nu{}_{\mu\alpha]}, \tag{33}$$

where each term on the right-hand side is zero because it antisymmetrizes something symmetric. Thus:

$$R^\gamma{}_{[\mu\alpha\beta]} = 0. \tag{34}$$

This eliminates $\frac{1}{6}n^2(n-1)(n-2) = 16$ additional components of the Riemann tensor, leaving $\frac{1}{3}n^2(n-1)(n+1) = 80$ components.

Since the Riemann tensor is already symmetric on the last two indices, a corollary to Eq. (34) is

$$R^\gamma{}_{\mu\alpha\beta} + R^\gamma{}_{\alpha\beta\mu} + R^\gamma{}_{\beta\mu\alpha} = 0. \tag{35}$$

If we lower the first index on the Riemann tensor, then more symmetries can be obtained. To see this, let's consider two vector fields $\boldsymbol{u}$ and $\boldsymbol{v}$. Then since $\boldsymbol{u}\cdot\boldsymbol{v}$ is a scalar, we have

$$(\boldsymbol{u}\cdot\boldsymbol{v})_{;\beta\alpha} = (\boldsymbol{u}\cdot\boldsymbol{v})_{;\alpha\beta}. \tag{36}$$

On the other hand, we could have written $\boldsymbol{u}\cdot\boldsymbol{v}$ using the metric tensor and taken covariant derivatives of everything. Remembering that the covariant derivative of the metric tensor is zero, we have

$$(g_{\mu\nu}u^\mu v^\nu)_{;\beta\alpha} = g_{\mu\nu}(u^\mu v^\nu)_{;\beta\alpha} = g_{\mu\nu}(u^\mu{}_{;\beta}v^\nu + u^\mu v^\nu{}_{;\beta})_{;\alpha} = g_{\mu\nu}(u^\mu{}_{;\beta\alpha}v^\nu + u^\mu{}_{;\beta}v^\nu{}_{;\alpha} + u^\mu{}_{;\alpha}v^\nu{}_{;\beta} + u^\mu v^\nu{}_{;\beta\alpha}). \tag{37}$$

Now let's subtract the same equation with α and β switched. The left-hand side then becomes zero, and on the right-hand side the 2nd and 3rd terms swap places and (after subtraction) leave us with zero. Then:

$$0 = g_{\mu\nu}[(u^\mu{}_{;\beta\alpha} - u^\mu{}_{;\alpha\beta})v^\nu + u^\mu(v^\nu{}_{;\beta\alpha} - v^\nu{}_{;\alpha\beta})]. \tag{38}$$

The right-hand side now contains the Riemann tensor:

$$0 = g_{\mu\nu}(R^\mu{}_{\kappa\alpha\beta}u^\kappa v^\nu + R^\nu{}_{\kappa\alpha\beta}u^\mu v^\kappa). \tag{39}$$

Lowering the indices on the Riemann tensor, and relabeling indices, then gives

$$0 = (R_{\nu\kappa\alpha\beta} + R_{\kappa\nu\alpha\beta})u^\kappa v^\nu, \tag{40}$$

and for this to be true for all $\boldsymbol{u}$ and $\boldsymbol{v}$, we require

$$R_{(\nu\kappa)\alpha\beta} = 0. \tag{41}$$

That is, the Riemann tensor is antisymmetric in its first two indices.

Yet another relation is implied by Eqs. (32,34,41). Let's consider the tensor defined by $K_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - R_{\gamma\delta\alpha\beta}$. Then

$$\begin{aligned} K_{\alpha\beta\gamma\delta} &= R_{\alpha\beta\gamma\delta} - R_{\gamma\delta\alpha\beta} \\ &= R_{\alpha\beta\gamma\delta} + R_{\gamma\alpha\beta\delta} + R_{\gamma\beta\delta\alpha} \\ &= -R_{\alpha\beta\delta\gamma} - R_{\alpha\gamma\beta\delta} + R_{\gamma\beta\delta\alpha} \\ &= R_{\alpha\delta\gamma\beta} - R_{\gamma\beta\alpha\delta} \\ &= -R_{\alpha\delta\beta\gamma} + R_{\beta\gamma\alpha\delta} \\ &= -K_{\alpha\delta\beta\gamma}. \end{aligned} \tag{42}$$

If we repeat this argument 3 times, we find

$$K_{\alpha\beta\gamma\delta} = -K_{\alpha\delta\beta\gamma} = K_{\alpha\gamma\delta\beta} = -K_{\alpha\beta\gamma\delta}, \tag{43}$$

so we must have $\mathbf{K} = 0$. This then implies that the Riemann tensor has the symmetry:

$$R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}. \tag{44}$$

This set of symmetries can be shown to reduce the Riemann tensor to only 20 independent components.

A final rule satisfied by the Riemann tensor concerns its covariant derivative. You will prove on the homework the *Bianchi identity*,

$$R_{\alpha\beta[\gamma\delta;\epsilon]} = 0. \tag{45}$$