Lecture VI: Tensor calculus

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(Dated: October 14, 2011)

I. OVERVIEW

In this lecture, we will begin with some examples from vector calculus, and then continue to define covariant derivatives of 1-forms and tensors.

The recommended reading for this lecture is:

- MTW §8.5–8.6.

[Note: actually a lot of what we're doing in class is working through the exercises in §8.5.]

II. A WORKED EXAMPLE: VECTOR CALCULUS IN POLAR COORDINATES

In this section, we will do some examples from vector calculus in polar coordinates on $\mathbb{R}^2$. This is a simple case, but should be useful to exercise the machinery. Recall that the metric tensor components were

$$g_{rr} = 1, \quad g_{\theta\theta} = r^2, \quad \text{and} \quad g_{r\theta} = g_{\theta r} = 0, \tag{1}$$

and the inverse metric is

$$g^{rr} = 1, \quad g^{\theta\theta} = \frac{1}{r^2}, \quad \text{and} \quad g^{r\theta} = g^{\theta r} = 0. \tag{2}$$

The coordinate basis vectors $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$ are not orthonormal, but we may define an orthonormal basis via

$$\boldsymbol{e}_{\hat r} = \boldsymbol{e}_r \quad \text{and} \quad \boldsymbol{e}_{\hat\theta} = \frac{1}{r}\boldsymbol{e}_\theta. \tag{3}$$

A. Christoffel symbols

We begin by computing the Christoffel symbols for polar coordinates. The only nonzero derivative of a covariant metric component is

$$g_{\theta\theta,r} = 2r. \tag{4}$$

Now returning to the general rule,

$$\Gamma^{\epsilon}{}_{\delta\eta} = \frac{1}{2} g^{\epsilon\tau}(-g_{\delta\eta,\tau} + g_{\eta\tau,\delta} + g_{\delta\tau,\eta}), \tag{5}$$

we can directly read off the Christoffel symbols. They are:

$$\begin{aligned}
\Gamma^r{}_{rr} &= 0, \\
\Gamma^r{}_{\theta r} = \Gamma^r{}_{r\theta} &= 0, \\
\Gamma^r{}_{\theta\theta} &= \frac{1}{2} g^{rr}(-g_{\theta\theta,r}) = \frac{1}{2}(1)(-2r) = -r, \\
\Gamma^\theta{}_{rr} &= 0, \\
\Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r} &= \frac{1}{2} g^{\theta\theta}(g_{\theta\theta,r}) = \frac{1}{2}(r^{-2})(2r) = \frac{1}{r}, \quad \text{and} \\
\Gamma^\theta{}_{\theta\theta} &= 0.
\end{aligned} \tag{6}$$

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So of the 6 Christoffel symbols, only 2 are nonzero. This is typical of highly symmetrical manifolds (expressed in coordinates that use the symmetry).

B. Covariant derivative of a vector field

Let's now find the covariant derivative of a vector field $\boldsymbol{v}$. Using the rule from the last section,

$$v^\alpha{}_{;\beta} = v^\alpha{}_{,\beta} + \Gamma^\alpha{}_{\beta\gamma} v^\gamma. \tag{7}$$

Component by component, this reads

$$\begin{aligned} v^r{}_{;r} &= v^r{}_{,r}, \\ v^r{}_{;\theta} &= v^r{}_{,\theta} - r v^\theta, \\ v^\theta{}_{;r} &= v^\theta{}_{,r} + \frac{1}{r} v^\theta, \quad \text{and} \\ v^\theta{}_{;\theta} &= v^\theta{}_{,\theta} + \frac{1}{r} v^r. \end{aligned} \tag{8}$$

The divergence of a vector field is

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = v^r{}_{;r} + v^\theta{}_{;\theta} = v^r{}_{,r} + v^\theta{}_{,\theta} + \frac{1}{r} v^r. \tag{9}$$

C. Examples of vector fields and their properties

Consider the vector field $\boldsymbol{v}$ that points in the 1-direction of the original Cartesian coordinate system ($\boldsymbol{v} = \boldsymbol{e}_1$). It can be expressed in terms of its components in the orthonormal basis

$$v^{\hat{r}} = \cos\theta, \quad v^{\hat{\theta}} = -\sin\theta; \tag{10}$$

or in the coordinate basis,

$$v^r = \cos\theta, \quad v^\theta = -\frac{1}{r}\sin\theta. \tag{11}$$

You know intuitively that this vector field is "constant," but that is not obvious in the polar coordinate system. We can still prove it, however, using Eq. (8):

$$\begin{aligned} v^r{}_{;r} &= v^r{}_{,r} = 0, \\ v^r{}_{;\theta} &= v^r{}_{,\theta} - r v^\theta = -\sin\theta - r(-\frac{1}{r}\sin\theta) = 0, \\ v^\theta{}_{;r} &= v^\theta{}_{,r} + \frac{1}{r} v^\theta = \frac{1}{r}^2 \sin\theta + \frac{1}{r}(-\frac{1}{r}\sin\theta) = 0, \quad \text{and} \\ v^\theta{}_{;\theta} &= v^\theta{}_{,\theta} + \frac{1}{r} v^r = -\frac{1}{r}\cos\theta + \frac{1}{r}\cos\theta = 0. \end{aligned} \tag{12}$$

As a less trivial example, we can search for an axisymmetric radial vector field $\boldsymbol{E}$ ($E^\theta = 0$, E^r depends only on r and not θ) with zero divergence (except at the origin): $\boldsymbol{\nabla} \cdot \boldsymbol{E} = 0$. Equation (9) tells us that we need

$$0 = E^r{}_{,r} + 0 + \frac{1}{r} E^r. \tag{13}$$

Since s^r depends on r, we may then write

$$\frac{d}{dr}(rE^r) = rE^r{}_{,r} + E^r = 0. \tag{14}$$

Therefore $E^{\hat{r}} = E^r \propto r^{-1}$. You may recognize this as the result that the electric field of a linear charge scales as $1/r$.

III. COVARIANT DERIVATIVES OF TENSORS

Having found a way to differentiate a vector (i.e. a rank $\binom{1}{0}$ tensor) in curved spacetime and generate a rank $\binom{1}{1}$ tensor, we now ask whether there is a way to differentiate general tensors in curved spacetime. The answer is yes, and fortunately there is no new messy algebra: the same Christoffel symbols we've worked with will suffice for tensors.

A. Covariant derivative of a 1-form

To warm up, let's try taking a 1-form – i.e. a rank $\binom{0}{1}$ tensor – and finding its covariant derivative, a rank $\binom{0}{2}$ tensor. We could repeat the work we did for vectors, going to a local Lorentz coordinate system, taking the derivatives, and going back to the original space. There is however an easier way: we may use the vector-to-1-form correspondence.

Given a 1-form k_μ, we know how to associate it with a vector $k^\alpha = g^{\alpha\mu}k_\mu$. Then we will find the covariant derivative of the 1-form by lowering the indices of $k^\alpha{}_{;\beta}$:

$$k_{\nu;\beta} \equiv g_{\nu\alpha}k^\alpha{}_{;\beta}. \tag{15}$$

Let's evaluate this explicitly:

$$\begin{aligned}
k_{\nu;\beta} &\equiv g_{\nu\alpha}k^\alpha{}_{;\beta} \\
&= g_{\nu\alpha}(g^{\alpha\mu}k_\mu)_{;\beta} \\
&= g_{\nu\alpha}[(g^{\alpha\mu}k_\mu)_{,\beta} + \Gamma^\alpha{}_{\beta\gamma}g^{\gamma\mu}k_\mu] \\
&= g_{\nu\alpha}\left[g^{\alpha\mu}{}_{,\beta}k_\mu + g^{\alpha\mu}k_{\mu,\beta} + \frac{1}{2}g^{\alpha\delta}(-g_{\beta\gamma,\delta} + g_{\beta\delta,\gamma} + g_{\gamma\delta,\beta})g^{\gamma\mu}k_\mu\right] \\
&= g_{\nu\alpha}g^{\alpha\mu}{}_{,\beta}k_\mu + k_{\nu,\beta} + \frac{1}{2}(-g_{\beta\gamma,\nu} + g_{\beta\nu,\gamma} + g_{\gamma\nu,\beta})g^{\gamma\mu}k_\mu.
\end{aligned} \tag{16}$$

As an aside at this point, we note that

$$g_{\nu\alpha}g^{\alpha\mu}{}_{,\beta} = (g_{\nu\alpha}g^{\alpha\mu})_{,\beta} - g_{\nu\alpha,\beta}g^{\alpha\mu} = (\delta^\mu_\nu)_{,\beta} - g_{\nu\gamma,\beta}g^{\gamma\mu} = -g_{\nu\gamma,\beta}g^{\gamma\mu}. \tag{17}$$

Therefore our covariant derivative of a 1-form satisfies

$$k_{\nu;\beta} = k_{\nu,\beta} + \frac{1}{2}(-g_{\beta\gamma,\nu} + g_{\beta\nu,\gamma} - g_{\gamma\nu,\beta})g^{\gamma\mu}k_\mu = k_{\nu,\beta} - \Gamma^\mu{}_{\beta\nu}k_\mu. \tag{18}$$

So we find the remarkable result that **the covariant derivative of a 1-form is given by the partial derivative, but corrected by a Christoffel symbol**. This is the same Christoffel symbol we found for vectors, but note the $-$ sign and the differently placed indices.

If we take the directional covariant derivative of a basis 1-form, we find

$$(\boldsymbol{\nabla}_{\boldsymbol{e}_\beta}\boldsymbol{\omega}^\alpha)_\nu = (\boldsymbol{\omega}^\alpha)_{\nu;\beta} = -\Gamma^\alpha{}_{\nu\beta}, \tag{19}$$

or equivalently

$$\boldsymbol{\nabla}_{\boldsymbol{e}_\beta}\boldsymbol{\omega}^\alpha = -\Gamma^\alpha{}_{\nu\beta}\boldsymbol{\omega}^\nu. \tag{20}$$

B. Covariant derivative of a general tensor

We are now ready to define the covariant derivative of a general tensor. Since we define covariant derivatives by reference to a local coordinate system where the covariant derivative becomes a partial derivative, it should satisfy a product rule. For example, if we take a tensor $\mathbf{S}$ of rank $\binom{2}{2}$, we have

$$\mathbf{S} = S^{\alpha\beta}{}_{\gamma\delta}\boldsymbol{e}_\alpha \otimes \boldsymbol{e}_\beta \otimes \boldsymbol{\omega}^\gamma \otimes \boldsymbol{\omega}^\delta. \tag{21}$$

To find the covariant derivative, we remember that

$$\boldsymbol{\nabla}_{\boldsymbol{e}_\mu}\boldsymbol{e}_\alpha = \Gamma^\nu{}_{\alpha\mu}\boldsymbol{e}_\nu. \quad \text{and} \quad \boldsymbol{\nabla}_{\boldsymbol{e}_\mu}\boldsymbol{\omega}^\alpha = -\Gamma^\alpha{}_{\nu\mu}\boldsymbol{\omega}^\nu, \tag{22}$$

so then

$$\begin{aligned}
\boldsymbol{\nabla}_{\boldsymbol{e}_\mu}\mathbf{S} &= S^{\alpha\beta}{}_{\gamma\delta,\mu}\boldsymbol{e}_\alpha\otimes\boldsymbol{e}_\beta\otimes\boldsymbol{\omega}^\gamma\otimes\boldsymbol{\omega}^\delta \\
&+S^{\alpha\beta}{}_{\gamma\delta}\boldsymbol{\nabla}_{\boldsymbol{e}_\mu}\boldsymbol{e}_\alpha\otimes\boldsymbol{e}_\beta\otimes\boldsymbol{\omega}^\gamma\otimes\boldsymbol{\omega}^\delta+S^{\alpha\beta}{}_{\gamma\delta}\boldsymbol{e}_\alpha\otimes\boldsymbol{\nabla}_{\boldsymbol{e}_\mu}\boldsymbol{e}_\beta\otimes\boldsymbol{\omega}^\gamma\otimes\boldsymbol{\omega}^\delta \\
&+S^{\alpha\beta}{}_{\gamma\delta}\boldsymbol{e}_\alpha\otimes\boldsymbol{e}_\beta\otimes\boldsymbol{\nabla}_{\boldsymbol{e}_\mu}\boldsymbol{\omega}^\gamma\otimes\boldsymbol{\omega}^\delta+S^{\alpha\beta}{}_{\gamma\delta}\boldsymbol{e}_\alpha\otimes\boldsymbol{e}_\beta\otimes\boldsymbol{\omega}^\gamma\otimes\boldsymbol{\nabla}_{\boldsymbol{e}_\mu}\boldsymbol{\omega}^\delta.
\end{aligned}\tag{23}$$

Plugging in the relations for the covariant derivatives of vectors and 1-forms, we find

$$\begin{aligned}
\boldsymbol{\nabla}_{\boldsymbol{e}_\mu}\mathbf{S} &= S^{\alpha\beta}{}_{\gamma\delta,\mu}\boldsymbol{e}_\alpha\otimes\boldsymbol{e}_\beta\otimes\boldsymbol{\omega}^\gamma\otimes\boldsymbol{\omega}^\delta \\
&+\Gamma^\nu{}_{\alpha\mu}S^{\alpha\beta}{}_{\gamma\delta}\boldsymbol{e}_\nu\boldsymbol{e}_\alpha\otimes\boldsymbol{e}_\beta\otimes\boldsymbol{\omega}^\gamma\otimes\boldsymbol{\omega}^\delta+\Gamma^\nu{}_{\beta\mu}S^{\alpha\beta}{}_{\gamma\delta}\boldsymbol{e}_\alpha\otimes\boldsymbol{e}_\nu\otimes\boldsymbol{\omega}^\gamma\otimes\boldsymbol{\omega}^\delta \\
&-\Gamma^\gamma{}_{\nu\mu}S^{\alpha\beta}{}_{\gamma\delta}\boldsymbol{e}_\alpha\otimes\boldsymbol{e}_\beta\otimes\boldsymbol{\omega}^\nu\otimes\boldsymbol{\omega}^\delta-\Gamma^\delta{}_{\nu\mu}S^{\alpha\beta}{}_{\gamma\delta}\boldsymbol{e}_\alpha\otimes\boldsymbol{e}_\beta\otimes\boldsymbol{\omega}^\gamma\otimes\boldsymbol{\omega}^\nu.
\end{aligned}\tag{24}$$

We may find the components of the right-hand side by relabeling indices:

$$\boldsymbol{\nabla}_{\boldsymbol{e}_\mu}\mathbf{S}=[S^{\alpha\beta}{}_{\gamma\delta,\mu}+\Gamma^\alpha{}_{\nu\mu}S^{\nu\beta}{}_{\gamma\delta}+\Gamma^\beta{}_{\nu\mu}S^{\alpha\nu}{}_{\gamma\delta}-\Gamma^\nu{}_{\gamma\mu}S^{\alpha\beta}{}_{\nu\delta}-\Gamma^\nu{}_{\delta\mu}S^{\alpha\beta}{}_{\gamma\nu}]\boldsymbol{e}_\alpha\otimes\boldsymbol{e}_\beta\otimes\boldsymbol{\omega}^\gamma\otimes\boldsymbol{\omega}^\delta\tag{25}$$

The object in brackets is then the ${}^{\alpha\beta}{}_{\gamma\delta\mu}$ component of the rank $\binom{2}{3}$ tensor $\boldsymbol{\nabla}\mathbf{S}$. Therefore it is the component of the covariant derivative of $\mathbf{S}$:

$$S^{\alpha\beta}{}_{\gamma\delta;\mu}=S^{\alpha\beta}{}_{\gamma\delta,\mu}+\Gamma^\alpha{}_{\nu\mu}S^{\nu\beta}{}_{\gamma\delta}+\Gamma^\beta{}_{\nu\mu}S^{\alpha\nu}{}_{\gamma\delta}-\Gamma^\nu{}_{\gamma\mu}S^{\alpha\beta}{}_{\nu\delta}-\Gamma^\nu{}_{\delta\mu}S^{\alpha\beta}{}_{\gamma\nu}.\tag{26}$$

The rules are quite general: one takes the partial derivative, and then writes down a correction term for each index. Each correction term is of the form "ΓS" and satisfies the following rules: (i) the differentiation index always appears in the last slot; (ii) the index being corrected moves over to Γ in either the up or down position as appropriate; (iii) one fills the two empty slots with a summed dummy index; and (iv) "up" indices get a $+$ sign and "down" indices get a $-$ sign. Covariant differentiation is somewhat mechanical, just like differentiation in freshman calculus.

C. Some more properties

There are a variety of useful properties of covariant derivatives that one can prove. Here is a sampling.

1. *The covariant derivative of the contraction is the same as the contraction of the covariant derivative*

In the local Lorentz coordinate frame established at any point $\mathcal{P}$, this is obvious. But let's try proving it explicitly: let's take a rank $\binom{1}{1}$ tensor $\mathbf{S}$ and find the contraction of its covariant derivative:

$$\begin{aligned}
S^\alpha{}_{\alpha;\beta}(\text{derivative then contraction}) &= S^\alpha{}_{\alpha,\beta}+\Gamma^\alpha{}_{\gamma\beta}S^\gamma{}_\alpha-\Gamma^\gamma{}_{\alpha\beta}S^\alpha{}_\gamma \\
&= S^\alpha{}_{\alpha,\beta}=(S^\alpha{}_\alpha)_{,\beta} \\
&= S^\alpha{}_{\alpha;\beta}(\text{contraction then derivative}).
\end{aligned}\tag{27}$$

The cancellation of the Γ-containing corrections is general for tensors of higher rank. So contraction and covariant derivative commute, and in the future we will simply write $S^\alpha{}_{\alpha;\beta}$ without ambiguity.

2. *The covariant derivative of the metric tensor is zero*

This is also obvious in the local Lorentz coordinate frame. But we can prove it explicitly:

$$\begin{aligned}
g_{\alpha\beta;\gamma} &= g_{\alpha\beta,\gamma}-\Gamma^\mu{}_{\alpha\gamma}g_{\mu\beta}-\Gamma^\mu{}_{\beta\gamma}g_{\alpha\mu} \\
&= g_{\alpha\beta,\gamma}-\frac{1}{2}g^{\mu\nu}(-g_{\alpha\gamma,\nu}+g_{\alpha\nu,\gamma}+g_{\gamma\nu,\alpha})g_{\mu\beta}-\frac{1}{2}g^{\mu\nu}(-g_{\beta\gamma,\nu}+g_{\beta\nu,\gamma}+g_{\gamma\nu,\beta})g_{\alpha\mu} \\
&= g_{\alpha\beta,\gamma}-\frac{1}{2}(-g_{\alpha\gamma,\beta}+g_{\alpha\beta,\gamma}+g_{\gamma\beta,\alpha})-\frac{1}{2}(-g_{\beta\gamma,\alpha}+g_{\beta\alpha,\gamma}+g_{\gamma\alpha,\beta}) \\
&= 0.
\end{aligned}\tag{28}$$

3. The covariant derivative of the inverse-metric tensor is zero

Once again: obvious in the local Lorentz coordinate frame. But there is a mathematical proof. Consider the rank $\binom{1}{1}$ "identity" tensor $\mathbf{I}$ that takes in a 1-form and vector and returns their contraction:

$$\mathbf{I}(\tilde{\boldsymbol{k}}, \boldsymbol{v}) = \langle \tilde{\boldsymbol{k}}, \boldsymbol{v} \rangle. \tag{29}$$

The components are easily seen to be $I^{\alpha}{}_{\beta} = \delta^{\alpha}{}_{\beta}$ (they are the same in any coordinate system!). We know that

$$I^{\alpha}{}_{\beta;\gamma} = \delta^{\alpha}{}_{\beta,\gamma} + \Gamma^{\alpha}{}_{\mu\gamma}\delta^{\mu}{}_{\beta} - \Gamma^{\mu}{}_{\beta\gamma}\delta^{\alpha}{}_{\mu} = 0. \tag{30}$$

But since $g^{\alpha\beta}$ is the matrix inverse of $g_{\alpha\beta}$, we have $I^{\alpha}{}_{\beta} = g^{\alpha\nu}g_{\nu\beta}$. Taking the covariant derivative, and recalling that covariant differentiation commutes with contraction:

$$I^{\alpha}{}_{\beta;\gamma} = g^{\alpha\nu}{}_{;\gamma}g_{\nu\beta} + g^{\alpha\nu}g_{\nu\beta;\gamma}. \tag{31}$$

The left-hand side is zero, $g_{\nu\beta;\gamma} = 0$, and so we are left with the conclusion that $g^{\alpha\nu}{}_{;\gamma}g_{\nu\beta} = 0$. Since $g_{\nu\beta}$ forms an invertible matrix, we can conclude that $g^{\alpha\nu}{}_{;\gamma} = 0$.

4. The covariant derivative commutes with raising and lowering of indices

Since the raising and lowering of indices involves the outer product with the metric tensor (or its inverse), followed by contraction, this statement is a corollary of the preceding ones.