Lecture IV: Stress-energy tensor and conservation of energy and momentum

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I. OVERVIEW

In this lecture, we will consider the spatial distribution of energy and momentum and their transport and conservation laws. The key new object that we will construct is the stress-energy tensor $T_{\mu\nu}$ – the right-hand side of Einstein's equation.

The recommended reading for this lecture is:

- MTW §5.1–5.7.

II. THE STRESS-ENERGY TENSOR

Consider a system containing matter, radiation, etc. observed in a particular Lorentz frame $\{\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$. We will study here the features of conservation of energy-momentum (a vector quantity). As a warmup, first it is worth recalling the conservation of (scalar) charge.

A. Charge

We learned in the lecture on E&M that one could construct a 4-vector J^α consisting of the charge density ($\rho = J^0$) and the current density (J^i). For a point particle, this is

$$J^\alpha = \int e u^\alpha \delta^{(4)}[x^\mu - y^\mu(\tau)]\, d\tau \tag{1}$$

and hence is Lorentz invariant; the "3+1" expression is

$$J^0(t, x^i) = e\delta^{(3)}[x^i - y^i(t)] \;\;\text{and}\;\; J^k(t, x^i) = e\, {}^{(3)}v^k \delta^{(3)}[x^i - y^i(t)]. \tag{2}$$

This was conserved in the local sense that

$$J^\alpha{}_{,\alpha} = \dot\rho + J^i{}_{,i} = 0. \tag{3}$$

Alternatively, if one tries to define a total charge

$$Q(t) = \int_{\mathbb{R}^3} \rho(t, x^i)\, d^3x^i \tag{4}$$

then its time derivative is

$$\dot Q(t) = \int_{\mathbb{R}^3} \dot\rho(t, x^i)\, d^3x^i = -\int_{\mathbb{R}^3} J^k{}_{,k}(t, x^i)\, d^3x^i = 0, \tag{5}$$

where the last statement is Gauss's divergence theorem (with boundary at infinity). So the total charge is globally conserved.

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B. Energy-momentum

We can repeat the same exercise for conservation of energy and momentum. Given a particle we may construct its 4-current density of 4-momentum, $T^{\beta\alpha}$:

$$T^{\beta\alpha} = \int p^\beta u^\alpha \delta^{(4)}[x^\mu - y^\mu(\tau)]\, d\tau. \tag{6}$$

It can be broken down into densities and fluxes of 4-momentum:

$$T^{\beta 0}(t,x^i) = p^\beta \delta^{(3)}[x^i - y^i(t)] \quad \text{and} \quad T^{\beta k}(t,x^i) = p^{\beta\,(3)} v^k \delta^{(3)}[x^i - y^i(t)]. \tag{7}$$

This tensor is called the *stress-energy tensor*. In "3+1" terminology, and in full generality (i.e. if we consider energy and momentum carried by fields as well as particles), the stress-energy tensor contains:

- The *energy density*: T^{00}.
- The *energy flux* in the i-direction: T^{0i}.
- The *3-momentum density*: T^{i0} (this is the density of momentum component i).
- The *3-momentum flux* (or "stress"): T^{ij} (this is the flux in the j direction of momentum component i).

The stress-energy tensor has 16 components, but we will see later that it is symmetric and only 10 are physical.

The usual statements about charge are equally valid for 4-momentum. It is conserved in the local sense that

$$T^{\beta\alpha}{}_{,\alpha} = \dot{T}^{\beta 0} + T^{\beta k}{}_{,k} = 0. \tag{8}$$

Alternatively, if one tries to define a total 4-momentum

$$P^\beta(t) = \int_{\mathbb{R}^3} T^{\beta 0}(t,x^i)\, d^3x^i \tag{9}$$

then its time derivative is

$$\dot{P}^\beta(t) = \int_{\mathbb{R}^3} \dot{T}^{\beta 0}(t,x^i)\, d^3x^i = -\int_{\mathbb{R}^3} T^{\beta k}{}_{,k}(t,x^i)\, d^3x^i = 0, \tag{10}$$

where again the last statement is Gauss's divergence theorem (with boundary at infinity). So the total 4-momentum is globally conserved.

III. EXAMPLES OF STRESS-ENERGY TENSORS

A. Example I: Perfect fluid

A perfect fluid in its rest frame has an energy density ρ and a pressure p. By isotropy, it carries zero energy flux and has zero momentum density. To compute the 3-momentum flux, we note that the flux of 1-momentum in the 1-direction is the momentum carried per unit time per unit area, i.e. the force per unit area p. This is the same as the flux of 2-momentum in the 2-direction, etc., so

$$T^{\beta\alpha} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}. \tag{11}$$

Equation (11) can be generalized to any frame by writing the right-hand side as a tensor. If the fluid has 4-velocity $\boldsymbol{u}$, then we find

$$T^{\beta\alpha} = \rho u^\beta u^\alpha + p(g^{\beta\alpha} + u^\beta u^\alpha). \tag{12}$$

This is in component notation; in terms of tensor operations it is

$$\mathbf{T} = \rho \boldsymbol{u} \otimes \boldsymbol{u} + p(\mathbf{g} + \boldsymbol{u} \otimes \boldsymbol{u}) = (\rho + p)\boldsymbol{u} \otimes \boldsymbol{u} + p\mathbf{g}. \tag{13}$$

B. Example II: Gas of noninteracting particles

Consider a gas composed of weakly or noninteracting identical particles each of mass m that do not form a perfect fluid (e.g. a real gas of atoms whose separation is many times the mean free path). They have a *phase space density* $f(x^i, p_i; t)$ which is the number of particles per unit volume in 3-position space per unit volume in 3-momentum space. The density of energy-momentum is

$$T^{\beta 0} = \int f(x^i, p_j; t)\, p^\beta \, d^3p_j \tag{14}$$

(valid **in a Lorentz frame**). The flux contains an additional factor of 3-velocity ${}^{(3)}v^k = u^k/u^0 = p^k/p^0$,

$$T^{\beta k} = \int f(x^i, p_j; t)\, p^\beta \, \frac{p^k}{p^0} \, d^3p_j. \tag{15}$$

These equations can be combined into

$$T^{\beta\alpha} = \int f(x^i, p_j; t)\, p^\beta p^\alpha \, \frac{d^3p_j}{p^0}. \tag{16}$$

A useful fact (and good homework exercise!) is to prove that $f(x^i, p_j; t)$ and d^3p_j/p^0 are individually Lorentz invariant. (In particle physics the latter sometimes goes by the name of "Lorentz invariant phase space.")

IV. SYMMETRY OF THE STRESS-ENERGY TENSOR

The stress-energy tensor must be symmetric. The standard way to prove this is to consider an infinitesimal cube of material of side length ℓ. What happens to it if the stress tensor is asymmetric, $T^{12} \neq T^{21}$? Then let's consider the 3-component of the torque on the cube. This comes from four faces, pointed in the $+1$, $+2$, -1, and -2 directions:

$$\begin{aligned} \tau_3 &= [\tau_3]_{+1} + [\tau_3]_{+2} + [\tau_3]_{-1} + [\tau_3]_{-2} \\ &= (-T^{21}\ell^2)(\ell/2) - (-T^{12}\ell^2)(\ell/2) + (T^{21})(-\ell/2) - (T^{12})(-\ell/2) \\ &= (T^{12} - T^{21})\ell^3. \end{aligned} \tag{17}$$

However the moment of inertia of the cube is $\frac{1}{6}\rho\ell^5$. Therefore in the limit that $\ell \to 0^+$, the cube undergoes an infinite angular acceleration unless $T^{21} = T^{12}$. Application of this argument in all reference frames guarantees a symmetric stress-energy tensor.

V. ENERGY OF THE ELECTROMAGNETIC FIELD

Not all energy-momentum is carried by particles. Some of it is associated with fields, and chief among these is the electromagnetic field $F_{\alpha\beta}$.

A. Construction of the stress-energy tensor

We may build the stress-energy tensor by considering first the energy density of the field. In undergraduate physics you learned that this was

$$\rho = \frac{1}{8\pi}(\boldsymbol{E}^2 + \boldsymbol{B}^2). \tag{18}$$

The challenge is to turn this into a full stress-energy tensor. We work in the frame of an observer with 4-velocity $\boldsymbol{u}$. The electric field seen by the observer is

$$E_\beta = F_{\beta\alpha} u^\alpha \tag{19}$$

(this has zero time component). So it follows that

$$\boldsymbol{E}^2 = F_{\beta\alpha} u^\alpha F^\beta{}_\gamma u^\gamma. \tag{20}$$

Furthermore, inspection of components of the field tensor shows that

$$F_{\delta\beta}F^{\delta\beta} = -2(\boldsymbol{E}^2 - \boldsymbol{B}^2) \tag{21}$$

and so

$$\boldsymbol{B}^2 = F_{\beta\alpha}u^\alpha F^\beta{}_\gamma u^\gamma + \frac{1}{2}F_{\delta\beta}F^{\delta\beta}. \tag{22}$$

Putting these together, and inserting a factor of $-g_{\alpha\gamma}u^\alpha u^\gamma = 1$, gives the energy density

$$\rho = \frac{1}{4\pi}\left(F_{\beta\alpha}F^\beta{}_\gamma - \frac{1}{4}F_{\delta\beta}F^{\delta\beta}g_{\alpha\gamma}\right)u^\alpha u^\gamma. \tag{23}$$

Since in general $\rho = T_{\alpha\gamma}u^\alpha u^\gamma$, and $\mathbf{T}$ is symmetric, the above relation can hold for all rest frames only if

$$T_{\alpha\gamma} = \frac{1}{4\pi}\left(F_{\beta\alpha}F^\beta{}_\gamma - \frac{1}{4}F_{\delta\beta}F^{\delta\beta}g_{\alpha\gamma}\right). \tag{24}$$

B. Implications

Equation (24), derived solely from the electromagnetic energy density, immediately implies several familiar facts from undergraduate physics.

Let us first consider the energy flux (Poynting flux) in say the 1-direction. This is the T^{01} component of the stress-energy tensor, or

$$T^{01} = \frac{1}{4\pi}F_\beta{}^0F^{\beta 1} = \frac{1}{4\pi}(F_2{}^0F^{21} + F_3{}^0F^{31}) = \frac{1}{4\pi}(E^2B^3 - E^3B^2). \tag{25}$$

If the energy flux forms a 3-vector S^i, then this is the 1-component of the equation

$$\boldsymbol{S} = \frac{1}{4\pi}\boldsymbol{E}\times\boldsymbol{B}. \tag{26}$$

Next let's consider the stress tensor. We'll consider the T^{11} component, i.e. the flux of 1-momentum carried in the 1-direction. This is

$$\begin{aligned} T^{11} &= \frac{1}{4\pi}\left(F_\beta{}^1F^{\beta 1} - \frac{1}{4}F_{\delta\beta}F^{\delta\beta}\right) \\ &= \frac{1}{4\pi}\left[-(E^1)^2 + (B^2)^2 + (B^3)^2 + \frac{1}{2}(\boldsymbol{E}^2 - \boldsymbol{B}^2)\right] \\ &= \frac{1}{8\pi}\left[(E^2)^2 + (E^3)^2 + (B^2)^2 + (B^3)^2 - (E^1)^2 - (B^1)^2\right]. \end{aligned} \tag{27}$$

So one can see that electric or magnetic fields exhibit positive stress (i.e. they "push") perpendicular to field lines, and exhibit negative stress (i.e. they "pull") parallel to field lines.