Ph 236 – Homework 9

Due: Monday, January 23, 2012

1. Higher symmetries. [12 points]

In class we considered spacetimes with Killing fields with the standard commutation relations of rotation

$$[\boldsymbol{J}_i, \boldsymbol{J}_j] = -\epsilon_{ijk}\boldsymbol{J}_k, \tag{1}$$

with two-dimensional orbits $\mathcal{S}(\mathcal{P})$. Cases where the orbits are higher-dimensional tend to be either trivial due to the high degree of symmetry, or very weird. Here we will consider a trivial example.

Consider the vectors in standard Minkowski spacetime given by

$$\boldsymbol{H}_i = \frac{1}{2}(\boldsymbol{J}_i + i\boldsymbol{K}_i) \tag{2}$$

(see Notes XVI, §IIIA for definitions of $\boldsymbol{J}_i$ and $\boldsymbol{K}_i$ in Minkowski).

(a) Show that

$$[\boldsymbol{H}_i, \boldsymbol{H}_j] = -\epsilon_{ijk}\boldsymbol{H}_k. \tag{3}$$

Thus the three fields described commute like rotations.

(b) Show that at a generic point $\mathcal{P}$, the three vectors $\boldsymbol{H}_1(\mathcal{P})$, $\boldsymbol{H}_2(\mathcal{P})$, and $\boldsymbol{H}_3(\mathcal{P})$ are linearly independent. Conclude that the orbit has dimension higher than 2.

(c) What symmetries do the Killing fields $\boldsymbol{H}_i$ correspond to? Construct an alternative (non-Minkowski) metric that has the $\boldsymbol{H}_i$ as Killing fields.

2. Spherical polytropes. [24 points]

Consider a star with a polytropic equation of state, $p = K\rho^\gamma$ where K and γ are constants.

(a) Show that the TOV equations reduce to a system of two ODEs for $\rho(r)$ and $m(r)$ as a function of radius.

(b) Now consider the particular case of $\gamma = 5/3$ and (through appropriate choice of units) $K = 1$. Explain why your system of equations can be initialized from $m(r_i) = 0$ and $\rho(r_i) = \rho_c$ at some very small inner radius r_i and integrated outward until $\rho = 0$ is reached.

(c) Using your favorite numerical integrator (in C, Fortran, Mathematica, ...) compute the total mass for several values of ρ_c. Show that there is a maximum mass; what is it?

(d) Rescaled to the equation of state of an ideal noninteracting Fermi-degenerate neutron gas (which is *not* a good approximation to the neutron star EOS, but is sufficient to illustrate the point), what is the predicted maximum mass?