Ph 236 – Homework 7
Due: Friday, December 2, 2011

1. More deflection of light by the Sun. [10 points]
If one is far enough from the Sun, a distant star directly behind the Sun can be gravitationally lensed into an *Einstein ring*.

(a) As a function of the distance D from the observer to the Sun and the mass M of the Sun, compute the radius $\theta_{\rm E}$ of the ring that one would expect to observe.

(b) In terms of D, M, and the radius R of the Sun, what is the minimum distance from the Sun for this effect to be observable? Numerically evaluate this result and express it in astronomical units.

2. Gravitational wave amplitudes. [12 points]
Determine the moment of inertia I, typical frequency ω, and hence the observed gravitational wave amplitude $h_{\rm GW}$ in the following situations. This is an order-of-magnitude problem; you may use "typical" astrophysical values and Kepler's third law.

(a) A "hot Jupiter" planet with a period of 4 days orbiting a solar-mass star, at a distance of 100 pc.

(b) Two neutron stars about to merge at a distance of 100 Mpc.

(c) A binary white dwarf with a period of 15 minutes at a distance of 10 kpc.

3. Searching for gravitational waves with pulsar timing. [24 points]
Consider a plane gravitational wave with metric:

$$ds^2 = -dt^2 + [1 + 2h_+ \cos\omega(t - x^3)]d(x^1)^2 + [1 - 2h_+ \cos\omega(t - x^3)]d(x^2)^2 + d(x^3)^2, \qquad (1)$$

with $|h_+| \ll 1$. Place the observer at $(0, 0, 0)$ and a pulsar at $(r_p \sin\theta\cos\phi, r_p \sin\theta\sin\phi, r_p \cos\theta)$. The pulsar emits pulses of radiation at times $t = nP$, where P is the pulsar period and n is an integer. Assume $P \ll \omega^{-1}$.

(a) Explain why the trajectories described for the observer and pulsar are geodesics.

(b) For a photon emitted with initial energy E in the nth pulse, write down the "unperturbed" (i.e. if $h_+ = 0$) trajectory from the pulsar to the observer. What is the arrival time at the observer?

(c) We now wish to determine the arrival time in the presence of the gravitational wave. The easiest way to do this is to perturb the null geodesic equation. Starting with $g_{\mu\nu}p^\mu p^\nu = 0$ show that if the metric is $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ then to first order in $h_{\mu\nu}$ one must have

$$h_{\mu\nu}p^\mu p^\nu + 2\eta_{\mu\nu}p^\mu \delta p^\nu = 0. \qquad (2)$$

Defining $\chi = \sin\theta\cos\phi\, x^1 + \sin\theta\sin\phi\, x^2 + \cos\theta\, x^3$, use this equation to find the formula for $d\chi/dt$ to first order in $h_{\mu\nu}$.

(d) Using the value of χ at the pulsar and at the observer, and using the results from (c), determine the arrival time of the nth pulse. Show that it is independent of the photon energy E, and that there is a gravitational wave-induced arrival time variation with frequency ω.

(e) What is the dependence of (d) on the position of the pulsar on the sky relative to the gravitational wave? Is the measurement polarization sensitive?