Ph 236 – Homework 5
Due: Friday, November 11, 2011

1. Cosmological constant. [16 points]
Consider the highly symmetrical spacetime with line element given by

$$ds^2 = \frac{1}{H^2\eta^2}(-d\eta^2 + dx^2 + dy^2 + dz^2), \tag{1}$$

where $H > 0$ is a constant, and in the domain $\eta < 0$. (This is known as *de Sitter spacetime*, and H is the Hubble constant.)

(a) Find the Einstein tensor for this spacetime.

(b) Prove that de Sitter spacetime is a vacuum solution of Einstein's equations in the presence of a positive cosmological constant for a particular choice of H. What is the relation between H and Λ?

(c) Consider an observer whose world line is given by fixed spatial coordinates (x, y, z). Explain why this trajectory is a geodesic, and show that an infinite amount of proper time elapses before the observer reaches $\eta = 0$.

2. Magnetic part of the Weyl tensor and its interpretation. [20 points]
Consider the magnetic part of the Weyl tensor, defined in class in any local Lorentz frame:

$$\mathcal{B}_{\hat{i}\hat{j}} = \frac{1}{2}\epsilon_{\hat{j}\hat{k}\hat{l}}C_{\hat{i}\hat{0}\hat{k}\hat{l}}. \tag{2}$$

(a) Prove that $\mathcal{B}_{\hat{i}\hat{j}}$ is symmetric and has zero trace, $\mathcal{B}_{\hat{i}\hat{i}} = 0$.

Now let us consider the problem of gravitationally induced precession of a gyroscope. Consider a family of observers traveling on timelike geodesics, parameterized by the proper time τ and a parameter n denoting the choice of geodesic (this is similar to our setup in deriving the geodesic deviation formula). Suppose that the 4-velocity of neighboring observers is $\boldsymbol{v} = \partial\mathcal{P}/\partial\tau$ and the infinitesimal separation of neighboring curves is $\xi = \partial\mathcal{P}/\partial n$. Suppose further that each observer carries a gyroscope whose angular momentum is pointed in the direction $\boldsymbol{\Omega}$. The gyroscopes are taken to be freely falling so $\boldsymbol{\Omega}$ is parallel-transported along each observer's trajectory – that is, $D\boldsymbol{\Omega}/\partial\tau = 0$.

(b) Prove that if $\boldsymbol{\Omega}$ is a purely spatial unit vector as seen by each observer at $\tau = 0$ – i.e. $|\boldsymbol{\Omega}(\tau = 0, n)|^2 = 1$ and $\boldsymbol{v} \cdot \boldsymbol{\Omega}(\tau = 0, n) = 0\ \forall n$ – then at any subsequent time these equations remain true.

Now let us define the infinitesimal difference of direction in which neighboring gyroscopes point,

$$\boldsymbol{\Delta} = \frac{D\boldsymbol{\Omega}}{\partial n}. \tag{3}$$

(c) Prove that

$$\frac{D\Delta^\alpha}{\partial\tau} = R^\alpha{}_{\beta\gamma\delta}\Omega^\beta v^\gamma \xi^\delta. \tag{4}$$

[*Hint*: The proof of this uses methodology similar to that for the geodesic deviation equation that we used in Lecture Notes VII §IIIB.]

Now specialize to a local Lorentz frame, and assume we are located in vacuum so that $R_{\mu\nu} = 0$.

(e) Prove that

$$\frac{D\Delta^{\hat{i}}}{\partial\tau} = C_{\hat{i}\hat{j}\hat{0}\hat{k}}\Omega^{\hat{j}}\xi^{\hat{k}}. \tag{5}$$

(f) Express the relevant component of the Weyl tensor in terms of the magnetic part $\mathcal{B}_{\hat{i}\hat{j}}$ and show that

$$\frac{D\Delta^{\hat{i}}}{\partial\tau} = \epsilon_{\hat{i}\hat{l}\hat{j}}\mathcal{B}_{\hat{l}\hat{k}}\xi^{\hat{k}}\Omega^{\hat{j}}. \tag{6}$$

This provides a physical interpretation of (and way to measure!) the magnetic part of the Weyl tensor: two sets of gyroscopes (inertial guidance systems) displaced by a separation ξ define reference frames that rotate relative to each other. The relative angular velocity of these reference frames is given by $\delta\omega_{\hat{l}} = \mathcal{B}_{\hat{l}\hat{k}}\xi^{\hat{k}}$.