1. **Cosmological constant.** [16 points]
Consider the highly symmetrical spacetime with line element given by

\[
ds^2 = \frac{1}{H^2\eta^2}(-d\eta^2 + dx^2 + dy^2 + dz^2),
\]

where \(H > 0\) is a constant, and in the domain \(\eta < 0\). (This is known as de Sitter spacetime, and \(H\) is the Hubble constant.)

(a) Find the Einstein tensor for this spacetime.

(b) Prove that de Sitter spacetime is a vacuum solution of Einstein’s equations in the presence of a positive cosmological constant for a particular choice of \(H\). What is the relation between \(H\) and \(\Lambda\)?

(c) Consider an observer whose world line is given by fixed spatial coordinates \((x, y, z)\). Explain why this trajectory is a geodesic, and show that an infinite amount of proper time elapses before the observer reaches \(\eta = 0\).

2. **Magnetic part of the Weyl tensor and its interpretation.** [20 points]
Consider the magnetic part of the Weyl tensor, defined in class in any local Lorentz frame:

\[
B^{\hat{i}\hat{j}} = \frac{1}{2}\epsilon^{\hat{k}\hat{l}}C_{\hat{i}\hat{j}\hat{k}\hat{l}}.
\]

(a) Prove that \(B^{\hat{i}\hat{j}}\) is symmetric and has zero trace, \(B^{\hat{i}\hat{i}} = 0\).

Now let us consider the problem of gravitationally induced precession of a gyroscope. Consider a family of observers traveling on timelike geodesics, parameterized by the proper time \(\tau\) and a parameter \(n\) denoting the choice of geodesic (this is similar to our setup in deriving the geodesic deviation formula). Suppose that the 4-velocity of neighboring observers is \(v = \partial \mathcal{P}/\partial \tau\) and the infinitesimal separation of neighboring curves is \(\xi = \partial \mathcal{P}/\partial n\). Suppose further that each observer carries a gyroscope whose angular momentum is pointed in the direction \(\Omega\). The gyroscopes are taken to be freely falling so \(\Omega\) is parallel-transported along each observer’s trajectory – that is, \(D\Omega/\partial \tau = 0\).

(b) Prove that if \(\Omega\) is a purely spatial unit vector as seen by each observer at \(\tau = 0\) – i.e. \(|\Omega(\tau = 0, n)|^2 = 1\) and \(v \cdot \Omega(\tau = 0, n) = 0 \forall n\) – then at any subsequent time these equations remain true.

Now let us define the infinitesimal difference of direction in which neighboring gyroscopes point,

\[
\Delta = \frac{D\Omega}{\partial n}.
\]

(c) Prove that

\[
\frac{D\Delta^\alpha}{\partial \tau} = R^\alpha_{\beta\gamma\delta} \Omega^\beta v^\gamma \xi^\delta.
\]

[Hint: The proof of this uses methodology similar to that for the geodesic deviation equation that we used in Lecture Notes VII §IIIB.]

Now specialize to a local Lorentz frame, and assume we are located in vacuum so that \(R_{\mu\nu} = 0\).

(e) Prove that

\[
\frac{D\Delta^i}{\partial \tau} = C_{ij\delta k} \Omega^j \xi^k.
\]
(f) Express the relevant component of the Weyl tensor in terms of the magnetic part $B_{ij}$ and show that

$$\frac{D\Delta^i}{\partial \tau} = \epsilon_{ijl} B_{ik} \xi^k \Omega^l. \quad (6)$$

This provides a physical interpretation of (and way to measure!) the magnetic part of the Weyl tensor: two sets of gyroscopes (inertial guidance systems) displaced by a separation $\xi$ define reference frames that rotate relative to each other. The relative angular velocity of these reference frames is given by $\delta \omega_i = B_{ik} \xi^k$. 