Ph 236 – Homework 4
Due: Friday, October 28, 2011

1. Surfaces of revolution. [14 points]
Consider a 2-dimensional surface $\mathcal{M}$ embedded in 3-dimensional Euclidean space $\mathbb{R}^3$. Suppose further that it is a surface of revolution, i.e. it can be described in cylindrical coordinates (z, ϖ, ϕ) by an equation of the form

$$z = f(\varpi). \tag{1}$$

The two-dimensional surface can be written locally with coordinates (ϖ, ϕ).

(a) Show that the line element for the surface is

$$ds^2 = F(\varpi)\, d\varpi^2 + \varpi^2\, d\phi^2, \tag{2}$$

where $F(\varpi) = 1 + [f'(\varpi)]^2$.

(b) Compute the Christoffel symbols for this surface.

(c) Compute the Riemann tensor R_{ijkl} for this surface (*hint*: use symmetries of the Riemann tensor to avoid having to do 16 tedious calculations).

(d) Show that the Ricci scalar is

$$R = \frac{F'(\varpi)}{\varpi[F(\varpi)]^2}. \tag{3}$$

(e) Prove that a 2-dimensional surface has zero Riemann tensor if and only if its Ricci scalar vanishes.

(f) Use (e) to completely classify the surfaces of revolution with zero Riemann tensor.

2. Geodesics and principle of least action in Newtonian gravity. [14 points]

Consider the following line element, which we will show later arises from slow-moving objects with nonrelativistic gravitational potentials:

$$ds^2 = -(1 + 2\Phi)dt^2 + (1 - 2\Phi)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2]. \tag{4}$$

Here the gravitational potential is a function of space and time, $\Phi(t, x^1, x^2, x^3)$, and satisfies $|\Phi| \ll 1$.

(a) Consider a slow-moving freely falling object, i.e. whose 3-velocity components $|v^i| \ll 1$. Explain why in this case $dt/d\tau \approx 1$ with an error of order Φ. Show that the i components of geodesic equation reduce to

$$\frac{d^2x^i}{dt^2} = -\Gamma^i{}_{00}. \tag{5}$$

(b) Compute the relevant Christoffel symbols and show that the apparent acceleration of an object is $-\nabla\Phi$.

(c) As an alternative description of the motion, consider the description of a geodesic as a world line of stationary proper time. Show that to second order in the velocity and first order in the potential, the proper time for a trajectory from $\mathcal{A} \to \mathcal{B}$ is

$$\Delta\tau = \int_{t(\mathcal{A})}^{t(\mathcal{B})} \left[1 - \frac{1}{2}\left(\frac{dx^1}{dt}\right)^2 - \frac{1}{2}\left(\frac{dx^2}{dt}\right)^2 - \frac{1}{2}\left(\frac{dx^3}{dt}\right)^2 + \Phi(t, x^1, x^2, x^3)\right] dt. \tag{6}$$

That is, the world line of *greatest* proper time is the path of *least* action.

(d) Suppose that Alice and Bob have identical ages and live on the surface of a planet where the acceleration due to gravity is g. Alice climbs into a rocketship that propels her upward from the surface at initial velocity v_0 (much less than the escape velocity, so that you can ignore effects resulting from changes in g as she climbs). She then goes into free fall, and after a time $t_0 = 2v_0/g$, Alice's rocketship lands back on her home planet, where Bob has been waiting. Who is older now, Alice or Bob, and by how much? Numerically evaluate this for $g = 9.8$ m/s^2 and $v_0 = 2$ km/s.

3. Derivative operations independent of the metric. [8 points]

The covariant derivative defined in class required the use of the metric tensor (or Christoffel symbols). However, a few types of derivative operation do not use the Christoffel symbols and hence make sense even without specifying the metric.

(a) Consider a 1-form field A_μ. Show that the 2-form $\boldsymbol{F} = \boldsymbol{dA}$ defined by

$$F_{\alpha\mu} \equiv A_{\mu;\alpha} - A_{\alpha;\mu} \tag{7}$$

can be written without Christoffel symbols:

$$F_{\alpha\mu} = A_{\mu,\alpha} - A_{\alpha,\mu}. \tag{8}$$

(b) Consider now the general p-form field $H_{\mu_1\ldots\mu_p}$. Let us define the $p+1$-form field $\boldsymbol{K} = \boldsymbol{dH}$ by

$$K_{\mu_1\ldots\mu_{p+1}} \equiv (p+1) H_{[\mu_1\ldots\mu_p;\mu_{p+1}]}. \tag{9}$$

Show that

$$K_{\mu_1\ldots\mu_{p+1}} = (p+1) H_{[\mu_1\ldots\mu_p,\mu_{p+1}]}. \tag{10}$$

(c) Consider two vector fields $\boldsymbol{u}$ and $\boldsymbol{v}$. Show that the vector field

$$w^\alpha = u^\beta v^\alpha{}_{;\beta} - v^\beta u^\alpha{}_{;\beta} \tag{11}$$

can be written as

$$w^\alpha = u^\beta v^\alpha{}_{,\beta} - v^\beta u^\alpha{}_{,\beta}. \tag{12}$$

The vector field $\boldsymbol{w}$ is called the "commutator" and is written as $\boldsymbol{w} = [\boldsymbol{u}, \boldsymbol{v}]$.