Ph 236 – Homework 3
Due: Friday, October 21, 2011

1. Uniqueness of the second-order terms in the coordinate transformation to a local Lorentz frame. [8 points]
In class, we claimed that once **A** was specified, there existed a unique value of $H_{\alpha'\beta'\epsilon'}$ [Eq. (33) of Lecture Notes V] that gave $g_{\alpha'\beta',\epsilon'}(0) = 0$. In class we proved existence by explicit construction of $H_{\alpha'\beta'\epsilon'}$. The purpose of this exercise is to prove uniqueness.

(a) Suppose there were two choices of $H_{\alpha'\beta'\epsilon'}$ – call them $[\mathbf{H}^{[A]}]_{\alpha'\beta'\epsilon'}$ and $[\mathbf{H}^{[B]}]_{\alpha'\beta'\epsilon'}$. Prove that the difference

$$\Delta_{\alpha'\beta'\epsilon'} \equiv [\mathbf{H}^{[A]}]_{\alpha'\beta'\epsilon'} - [\mathbf{H}^{[B]}]_{\alpha'\beta'\epsilon'} \tag{1}$$

is symmetric on the last two indices ($\Delta_{\alpha'\beta'\epsilon'} = \Delta_{\alpha'\epsilon'\beta'}$) and antisymmetric on the first two indices ($\Delta_{\alpha'\beta'\epsilon'} = -\Delta_{\beta'\alpha'\epsilon'}$).

(b) Prove that any third-rank tensor symmetric on the last two indices and antisymmetric on the first two indices must equal zero.

2. Vector calculus in spherical coordinates. [16 points]
You have probably done lots of vector calculus work in $\mathbb{R}^3$ using the spherical coordinate system. Here we will do all of that, but using the powerful new methods from this week's class. The spherical coordinates (r, θ, ϕ) are to be related to the usual Cartesian coordinates via

$$x^1 = r\sin\theta\cos\phi, \quad x^2 = r\sin\theta\sin\phi, \quad \text{and} \quad x^3 = r\cos\theta. \tag{2}$$

(a) Express the spherical coordinate basis vectors $\boldsymbol{e}_r$, $\boldsymbol{e}_\theta$, and $\boldsymbol{e}_\phi$ in terms of the Cartesian basis vectors. Show that they are orthogonal but have varying norms.

(b) Write down the metric tensor and line element in spherical coordinates.

(c) Construct the 18 Christoffel symbols in spherical coordinates.

(d) Given a vector field $\boldsymbol{v}$ described by its components (v^r, v^θ, v^ϕ), write a formula for the divergence $\nabla \cdot \boldsymbol{v}$.

(e) Commonly one uses the orthonormal (non-coordinate) basis $\{\boldsymbol{e}_{\hat{r}}, \boldsymbol{e}_{\hat{\theta}}, \boldsymbol{e}_{\hat{\phi}}\}$, defined by taking the coordinate basis vectors and normalizing them (as noted in class, this step only works for orthogonal coordinates). In terms of these normalized components $(v^{\hat{r}}, v^{\hat{\theta}}, v^{\hat{\phi}})$, what is the divergence $\nabla \cdot \boldsymbol{v}$?

3. Vector calculus identities. [12 points]
Prove the following identifies regarding derivatives of scalar and vector fields.

(a) The scalar product rule: for scalar field f and vector field $\boldsymbol{v}$,

$$(f\boldsymbol{v})^\alpha{}_{;\beta} = f_{,\beta}v^\alpha + fv^\alpha{}_{;\beta}. \tag{3}$$

(b) The dot product rule: for vector fields $\boldsymbol{v}$ and $\boldsymbol{w}$,

$$(\boldsymbol{v} \cdot \boldsymbol{w})_{,\alpha} = v_\beta w^\beta{}_{;\alpha} + w_\beta v^\beta{}_{;\alpha}. \tag{4}$$

(c) The simple formula for the divergence of a vector field expressed in the coordinate basis,

$$\nabla \cdot \boldsymbol{v} = v^\alpha{}_{,\alpha} + \frac{1}{2}g^{\alpha\mu}g_{\alpha\mu,\beta}v^\beta. \tag{5}$$

[Note: this can be further simplified to $\nabla \cdot \boldsymbol{v} = v^\alpha{}_{,\alpha} + (\ln\sqrt{\det \mathbf{g}})_{,\beta}v^\beta$, where $\det \mathbf{g}$ is the determinant of the metric tensor in $n \times n$ matrix form, but proving this is not part of the homework.]