Ph 236 – Homework 18
Due: Friday, May 4, 2012

1. Properties of the electromagnetic field. [18 points]
Consider a free electromagnetic field generated by a 1-form potential $\boldsymbol{A}$ by the usual rule $\boldsymbol{F} = \boldsymbol{dA}$. The electromagnetic field is gauge-invariant in the sense that one can add the gradient of a scalar χ to the potential, $\boldsymbol{A} \rightarrow \boldsymbol{A} + \boldsymbol{d}\chi$, with no effect on the field $\boldsymbol{F}$.

We will take the 4 components $A_\mu(x^\alpha)$ as the quantities to be varied in the variational principle $\delta S = 0$. The Lagrange density for the system is of the form

$$\mathcal{L} = \mathcal{L}_{\rm GR} + \mathcal{L}_{\rm EM} + \mathcal{L}_{\rm other\ matter} + \mathcal{L}_{\rm int}, \tag{1}$$

where $\mathcal{L}_{\rm EM}$ depends only on the electromagnetic field and the metric, $\mathcal{L}_{\rm other\ matter}$ does not depend on the electromagnetic field, and all terms that depend on both (i.e. describe interactions of electromagnetic fields with other matter) are included in $\mathcal{L}_{\rm int}$.

(a) Suppose we take the usual electromagnetic Lagrange density $\mathcal{L}_{\rm EM} = -\frac{1}{16\pi} F_{\mu\nu}F^{\mu\nu}$. Express the action explicitly in terms of A_μ and the metric, and show that its contribution to the stress-energy tensor is

$$[T^{\rm rel}_{\rm EM}]^{\mu\nu} = 2(-g)^{-1/2} \left.\frac{\delta S_{\rm EM}}{\delta g_{\mu\nu}}\right|_{A_\nu} \tag{2}$$

is the usual stress-energy tensor.

(b) Now suppose that the interaction Lagrangian is of the form

$$\mathcal{L}_{\rm int} = A_\mu J^\mu[\psi], \tag{3}$$

where J^μ is called the *current density* and it depends on other matter fields ψ (e.g. point particles, charged scalar fields, or anything else, but not explicitly on $\boldsymbol{A}$). What is the equation of motion for A_μ?

(c) Now let us require that the total action be gauge invariant for arbitrary gauge transformation field χ. What condition does this imply about the source $\boldsymbol{J}$?

2. Dynamics of theories with higher derivatives in the action. [18 points]
In this problem, we will investigate theories in which the Lagrangian depends on the accelerations of particles in addition to their coordinates and velocities.

(a) Consider a system with N degrees of freedom and a Lagrangian of the form

$$L = L_{(1)}(q^I, \dot q^I) + L_{(2)A}(q^I)\ddot q^A \tag{4}$$

(here A is summed over $1..N$). Show that the variation of the action $\delta S/\delta q_A(t)$ can be manipulated by integration by parts, and thus reduced to a form containing only coordinates and velocities, and not accelerations.

Thus for "generic" functions $L_{(1)}$ and $L_{(2)}$ one arrives at a system of second-order ODEs for the q_A, just as in usual Newtonian physics. I impose the "generic" issue since there is a special class of functions $L_{(1)}$ and $L_{(2)}$ for which this construction does not work – e.g. if we write a Lagrangian where there is no dependence on the derivatives, but is still technically of the form Eq. (4).

(b) In order for the equations to yield a second-order system, i.e. to find $\ddot q^A$ as functions of $\{q^I, \dot q^I\}$, the system must satisfy a non-singularity condition. Show that this condition is that the $N \times N$ symmetric matrix $\mathbf{C}$ defined by

$$C_{IJ} = \frac{\partial^2 L_{(1)}}{\partial \dot q^I\, \partial \dot q^J} - \frac{\partial L_{(2)J}}{\partial q^I} - \frac{\partial L_{(2)I}}{\partial q^J} \tag{5}$$

be nonsingular. [*Hint*: Write down the conjugate momenta p_I derived from the construction in part (a).] When we do this construction in GR, it will turn out that this non-singularity condition is **not** satisfied, and hence it is not possible to solve for the full evolution $g_{\mu\nu}(x^\alpha)$ given a set of initial conditions; this is related to gauge degrees of freedom, which must be fixed externally and cannot be "solved for."

(c) Now let us write down a Lagrangian nonlinear in the accelerations. An illustrative possibility is

$$L = L_{(1)}(q^I, \dot{q}^I) + \frac{1}{2} L_{(2)AB}(q^I) \ddot{q}^A \ddot{q}^B, \tag{6}$$

where $L_{(2)AB}$ is symmetric. Show that the Euler-Lagrange equations generically lead to a fourth-order system of ODEs. What is the non-singularity condition required for us to solve for $\partial_t^4 q^A$ in terms of the coordinates and the first 3 derivatives?