Ph 236 – Homework 17
Due: Friday, April 27, 2012

1. Variational approach to universes with matter. [20 points]
This problem works through the Lagrangian derivation of the Friedmann equations, and then considers some issues associated with the "total energy of the Universe" in cosmology. When we consider the variational principle, we will focus only on variations that respect the FRW symmetry (inhomogeneous universes will be considered later).

We will show next week that the action for general relativity is the Einstein-Hilbert action,

$$S_{\rm EH} = \int \frac{R}{16\pi}\sqrt{-g}\,d^4x, \tag{1}$$

where g denotes the determinant of the metric tensor, and R is the Ricci scalar.

(a) Show that $\sqrt{-g}\,d^4x$ is the differential proper 4-volume element.

(b) Using the Ricci scalar from the notes, show that for an FRW universe, if we consider the comoving volume $\mathcal{V}$, then the action may be written as an integral containing $\ddot{a}$. Reduce this to a form containing at most first time derivatives of a using integration by parts and throwing out surface terms to show that

$$S_{\rm EH} = \frac{3}{8\pi}\mathcal{V}\int \left[-a\dot{a}^2 + aK\right]\,dt. \tag{2}$$

(c) Recall that the action for a point particle of mass μ is $S_{\rm part} = -\mu\int d\tau$, where $d\tau$ is the differential of proper time. Show that if these particles are at rest in the comoving frame, and there is a comoving density ρ_0, then this action reduces to

$$S_{\rm part} = -\rho_0\mathcal{V}\int dt. \tag{3}$$

We therefore conclude that in a universe with cold noninteracting particles, the total action is

$$S = \mathcal{V}\int\left[-\frac{3}{8\pi}a\dot{a}^2 + \frac{3K}{8\pi}a - \rho_0\right]\,dt. \tag{4}$$

(d) Show that if this action is varied with fixed t_1, t_2, $a(t_1)$, and $a(t_2)$, allowing the trajectory of a to vary in between the initial and final times, that the Euler-Lagrange equation gives the **second-order equation**:

$$\frac{\ddot{a}}{a} = -\frac{1}{2}\left(\frac{\dot{a}}{a}\right)^2 - \frac{K}{2a^2}. \tag{5}$$

Show that the Friedmann equations imply Eq. (5), **but** that Eq. (5) does not imply the Friedmann equations.

You will note that in part (d) we did not allow t_1 or t_2 to vary. In general relativity – unlike Newtonian physics or even special relativity – the action should be stationary with respect to variations where we keep the initial and final states fixed but allow arbitrary behavior in between, including changing the total proper time seen by the matter particles, which in the FRW coordinates is $t_2 - t_1$. Therefore, in part (d), we did not allow the most general legal variation. We should have fixed the initial and final a and allowed t_1 and t_2 to float.

(e) Re-write Eq. (4) in terms of the function $t(a)$. Allowing general variations with t_1 and t_2 free but fixing a_1 and a_2, show that you can derive an equation involving $t(a)$. Re-writing it in terms of $a(t)$, show that you arrive at the **first-order equation**:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi\rho_0}{3a^3} + \frac{K}{a^2}. \tag{6}$$

This is an example of a *constraint equation* – a constraint on initial conditions (i.e. on a field and its first derivative) due to the existence of gauge degrees of freedom (in this case choosing the labeling of surfaces of constant t). The fact that we have fixed the gauge by setting $g_{tt} = -1$ does not remove the constraint!

(f) In any system of the form $S = \int L(a, \dot{a})\, dt$, where L is a Lagrangian with no explicit time dependence, it is possible to construct a conserved Hamiltonian H. In ordinary mechanics (including special relativity), H is identified with the total energy. Construct H for the FRW universe and show that it is always equal to zero.

2. Features in the cosmic microwave background. [16 points]
This problem concerns itself with the angular scale of features in the CMB due to sound waves in the early universe. These waves are sourced by primordial density perturbations, and propagate in the plasma of baryons and photons until the time of recombination (when the cosmic plasma cools, becomes neutral, and hence transparent to radiation that no longer Thomson scatters).

(a) If the primordial plasma has $\rho_\gamma \gg \rho_{\rm b}$, then the photon-baryon plasma has $w = \frac{1}{3}$. Show that the sound speed in such a plasma is $c_{\rm s} = 1/\sqrt{3}$.

(b) Find the scale factor $a_{\rm rec}$ at recombination in terms of the temperature $T_{\rm rec}$ of recombination (typically ~ 3000 K) and the CMB temperature T_0 today.

(c) Find the comoving distance $r_{\rm s}$ that a sound wave travels between the Big Bang and recombination. For this part, you may assume that during the epoch of interest $\rho \approx \rho_{\rm m}$ (radiation is actually important in this calculation at the level of a few tens of percents, but this assumption is good enough to illustrate the physical principle involved), and leave your answer in terms of $T_{\rm rec}$, T_0, $\Omega_{\rm m}$, and H_0.

(d) In the limit $T_0 \ll T_{\rm rec}$, find the comoving angular diameter distance to the epoch of recombination in terms of $T_{\rm rec}$, T_0, $\Omega_{\rm m}$, and H_0. Assume (incorrectly) that $\Lambda = 0$.

(e) Show that the angular size of the sound propagation distance, $\theta_{\rm s}$, is a function of $\Omega_{\rm m}$ and $T_{\rm rec}/T_0$. What is this function?

The answer to (e), combined with the observations of $\theta_{\rm s}$ from the CMB around 2000, were a key line of evidence (maybe **the** most important line of evidence) that led cosmologists to abandon the open $\Omega_{\rm m} \sim 0.2$, $\Lambda = 0$ cosmological model.