**Ph 236 – Homework 16**
Due: Friday, April 20, 2012

**1. Geometry of hyperbolic space.** [20 points]
Consider the hyperbolic space $\mathbb{H}^3$ with unit negative curvature,

$$ds^2 = d\chi^2 + \sinh^2\chi\,(d\theta^2 + \sin^2\theta\,d\phi^2). \tag{1}$$

(a) Show that this is the geometry that one obtains by taking 4-dimensional Minkowski space $\mathbb{M}^4$ and restricting to the spacelike 3-surface $\Sigma \subset \mathbb{M}^4$ given by

$$-t^2 + x^2 + y^2 + z^2 = -1. \tag{2}$$

What is the mapping $\Phi : \mathbb{H}^3 \to \Sigma$ of the coordinates $(\chi, \theta, \phi)$ to $(t, x, y, z)$?

(b) Using your knowledge of infinitesimal Lorentz transformations, find the 6 Killing fields corresponding to Lorentz rotations and boosts, and express them as contravariant vectors in the $\chi\theta\phi$ coordinate system.

(c) Show that given two points $\mathcal{P}, \mathcal{Q} \in \mathbb{H}^3$ that the separation $s$ between them (i.e. length of the shortest[1] geodesic that connects them) is given by

$$\cosh s = -\Phi(\mathcal{P}) \cdot \Phi(\mathcal{Q}), \tag{3}$$

where the dot product is that defined in $\mathbb{M}^4$.

(d) Use the result of (c) to prove that in the hyperbolic space, the law of cosines is given by

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos\gamma, \tag{4}$$

where we denote the side lengths of a "triangle" (i.e. set of 3 geodesics connecting the 3 vertices) as $a$, $b$, and $c$, and the angles opposite them as $\alpha$, $\beta$, and $\gamma$.

(e) Taylor expand the result from (d) and show that if $a$, $b$, and $c$ are $\ll 1$ then the usual law of cosines is recovered.

**2. Neutrinos in an expanding universe.** [16 points]
Consider an expanding, spatially flat FRW universe that contains a gas of neutrinos and anti-neutrinos. At some epoch with scale factor $a = a_i \ll 1$, assume that the neutrinos are distributed according to a fermionic blackbody at temperature $T_i$ with no lepton asymmetry, i.e. the occupation number is

$$f({}^{(3)}\boldsymbol{p}) = \frac{1}{\exp\left(\sqrt{|{}^{(3)}\boldsymbol{p}|^2 + m_\nu^2}\,/T_i\right) + 1}. \tag{5}$$

Here ${}^{(3)}\boldsymbol{p}$ is the 3-momentum measured by a comoving observer and $m_\nu$ is the mass of the neutrino (or the neutrino species in question).

Thereafter imagine that neutrino interactions are turned off and they behave as test particles. (This is actually a good approximation in the early Universe; the switch-off occurs at $t$ of order a second when the density drops low enough that the neutrino interaction timescale exceeds the age of the Universe. We will treat only one neutrino species in this problem.) You may assume $T_i \gg m_\nu$, so that at $a = a_i$ all the neutrinos can be assumed highly relativistic. However we are interested in what happens at much later times $a \gg a_i$, where some of the neutrinos may be mildly relativistic or nonrelativistic.

[1] In hyperbolic space, the statement about the shortest geodesic is redundant: there is exactly one geodesic connecting any two points. This is actually true for any simply connected space of Euclidean signature $(+++)$ and negative-semidefinite curvature, i.e. where $R_{\alpha\beta\gamma\delta}\xi^\alpha\eta^\beta\xi^\gamma\eta^\delta \le 0$ for any vectors $\xi$ and $\eta$.

(a) Show that the 3-momentum ${}^{(3)}\boldsymbol{p}$ of a neutrino decays as ${}^{(3)}\boldsymbol{p} \propto 1/a$. Explain why neutrino masses become significant at a scale factor $a \sim a_{\rm nr} = a_i T_i / m_\nu$.

(b) Write down the integrals for the density $\rho_\nu$ and pressure $p_\nu$ of neutrinos as a function of scale factor $a$. Make sure your integrals are valid whether $a$ is smaller than, greater than, or of order $a_{\rm nr}$.

(c) Prove analytically that the density and pressure you constructed obey the continuity equation.

(d) Show that at $a \ll a_{\rm nr}$ you have $\rho_\nu \propto a^{-4}$, whereas at $a \gg a_{\rm nr}$ you have $\rho_\nu \propto a^{-3}$.