Ph 236 – Homework 14
Due: Friday, March 2, 2012

1. Entropy of a black hole. [18 points]
Suppose that we imagine that the "temperature" T of a black hole could be associated with a set of internal degrees of freedom of the hole itself. In this case, one should be able to write down the entropy $\mathcal{S}$ in accordance with the first law of thermodynamics. For the purposes of the analytical formulae, you may set $\hbar = 1$ in addition to $G = c = 1$.

(a) For a Schwarzschild hole, the first law of thermodynamics reads

$$dM = T\,d\mathcal{S}. \tag{1}$$

For a Schwarzschild black hole, solve for d$\mathcal{S}$ (assuming $\mathcal{S} \to 0$ as $M \to 0$) and then evaluate the entropy as a function of mass M. What is the entropy of a $10M_\odot$ black hole (in conventional units with Boltzmann's constant equal to 1)?

(b) Now let us consider a charged nonrotating hole. Explain why the temperature is now

$$T = \frac{1}{8\pi K}, \tag{2}$$

where K is defined as in Lecture Notes XXV. [*Hint*: You should not need to do a complicated calculation for this part.]

(c) The first law of thermodynamics now reads

$$dM = T\,d\mathcal{S} + U\,dQ, \tag{3}$$

where U is the electric potential of the hole, which (based on the conservation law for energy of a charged particle) we would naturally identify with $-A_t$ at the horizon.[1] Show that there exists an entropy function $\mathcal{S}$ if and only if the *Maxwell relation* holds:

$$\left.\frac{\partial K}{\partial Q}\right|_M = -\left.\frac{\partial (KU)}{\partial M}\right|_Q. \tag{4}$$

[The Maxwell relation is thus a consistency test for treating a charged black hole as a statistical-mechanical system.] Explicitly verify that the Maxwell relation holds.

(d) Having established in (c) that $\mathcal{S}$ exists, you may integrate the first law of thermodynamics along any path to evaluate it. Show that

$$\mathcal{S} = \pi r_+^2. \tag{5}$$

That is, **the entropy is equal to $\frac{1}{4}$ of the horizon area**. Thus the rule that the area of the horizon increases is really a restatement of the second law of thermodynamics.

2. **The de Sitter horizon**. [18 points]
Consider again the Tolman-Oppenheimer-Volkoff metric,

$$ds^2 = -e^{2\Phi}\,dt^2 + \frac{dr^2}{1-2m/r} + r^2(d\theta^2 + \sin^2\theta\,d\phi^2). \tag{6}$$

(a) Explain why adding a cosmological constant to Einstein's equations results in replacing $\rho \to \rho + \Lambda/(8\pi)$ and $p \to p - \Lambda/(8\pi)$ in the TOV equations.[2]

[1] This statement should be understood as in a gauge in which $\mathcal{L}_\xi \mathbf{A} = 0$, where ξ is the timelike Killing field. This includes the gauge used in class.

[2] I apologize for the standard use of Λ in the derivation of the TOV equations – please be sure to avoid confusion.

(b) Show that in the presence of a cosmological constant, in vacuum the dp/dr TOV equation is trivially satisfied. Then assuming the metric to be well-behaved at the origin, solve for $m(r)$ and $\Phi(r)$, and show that the metric can be cast in the form

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta\,d\phi^2), \tag{7}$$

where $f(r) = 1 - \Lambda r^2/3$. This spacetime is called *de Sitter spacetime.*

(c) Show that there exists a coordinate singularity at a radius coordinate $r = r_{\rm dS} = \sqrt{3/\Lambda}$. This surface has the same mathematical structure as the event horizon of Schwarzschild or Reissner-Nordstrøm, and hence we expect it to act as an event horizon, except "inside out."

(d) Show that you can define a re-scaled coordinate $r_\star(r)$ to cast the metric in the form

$$ds^2 = f(r)\,(-dt^2 + dr_\star^2) + r^2(d\theta^2 + \sin^2\theta\,d\phi^2) \tag{8}$$

by defining

$$r_\star = \frac{r_{\rm dS}}{2}\ln\frac{r_{\rm dS} + r}{r_{\rm dS} - r}, \tag{9}$$

which is zero at the origin and stretches to $r_\star = +\infty$ at $r = r_{\rm dS}$. Looking at the coefficient of $\ln(r_{\rm dS} - r)$, argue by analogy to our calculation for the Schwarzschild and Reissner-Nordstrøm cases that the horizon radiates at an apparent temeprature of

$$T = \frac{1}{2\pi r_{\rm dS}}. \tag{10}$$

[Note: if the true explanation for the accelerating Universe is indeed Λ, then in the far future the observable Universe approaches the de Sitter state, and all of the distant galaxies fall through the de Sitter horizon and exponentially fade away.]