Ph 236 – Homework 13
Due: Friday, February 24, 2012

1. Orders of magnitude. [8 points]
Answer the following questions to an order of magnitude.

(a) What is the maximum mass of a black hole that can evaporate via Hawking radiation in the lifetime of the Universe?

(b) For a $10M_\odot$ black hole, what is the charge in conventional units (Coulombs) that would have to be present to make $Q = M$ in relativistic units? How does this compare to the total charge of $10M_\odot$ of protons?

2. Electrostatics near a black hole – Part I. [14 points]
Often theoretical astrophysicists who don't want to solve the whole system of equations treat a black hole horizon as if it were a conducting sphere (usually with some surface resistivity). This problem works through the reasons for this.

Imagine that we take a Schwarzschild black hole of mass M, and construct a system of charges that is static (independent of time) and axisymmetric (just for simplicity). This system will have charge density $\rho(r,\theta)$ and zero current density seen by a stationary observer, i.e. a stationary observer at coordinate radius r with 4-velocity $\boldsymbol{u}$ sees $\boldsymbol{J} = \rho\boldsymbol{u}$, where $\boldsymbol{J}$ is the 4-current density. We will also assume a static electromagnetic field, i.e. if ξ is the timelike Killing field, then $\mathcal{L}_\xi F_{\mu\nu} = 0$. Throughout, assume the electromagnetic fields are weak enough that their stress-energy tensor is negligible.

(a) Use symmetry arguments to show that the stationary observer using the coordinate basis sees only field components $E_{\hat r}$ and $E_{\hat\theta}$ – that is, explain why $E_{\hat\phi} = B_{\hat r} = B_{\hat\theta} = B_{\hat\phi} = 0$. Write down the nonvanishing covariant components $F_{\mu\nu}$ in terms of $\boldsymbol{E}$ and $\boldsymbol{B}$ in the oberver's orthonormal frame.

(b) Show that Maxwell's equations imply

$$\partial_\theta E_{\hat r} - \partial_r\left(r\sqrt{1-2\frac{M}{r}}\,E_{\hat\theta}\right) = 0 \tag{1}$$

and

$$\partial_r E_{\hat r} + \frac{2}{r}E_{\hat r} + \frac{\partial_\theta E_{\hat\theta}}{r\sqrt{1-2M/r}} + \frac{\cot\theta}{r\sqrt{1-2M/r}}E_{\hat\theta} = 4\pi\frac{\rho}{\sqrt{1-2M/r}}. \tag{2}$$

Which component of which Maxwell equation gives rise to each?

(c) Show that the first of Maxwell's equations implies the existence of a potential Φ with

$$E_{\hat r} = -\partial_r\Phi \quad \text{and} \quad E_{\hat\theta} = -\frac{1}{r\sqrt{1-2M/r}}\partial_\theta\Phi. \tag{3}$$

By comparison to your answer to part (a), show that $A_\mu = (-\Phi, 0, 0, 0)$ is a vector potential that generates the electromagnetic field via $\boldsymbol{F} = \boldsymbol{dA}$.

(d) Combining your results for Maxwell's equations to yield

$$-\frac{\sqrt{1-2M/r}}{r^2}\partial_r(r^2\partial_r\Phi) - \frac{1}{r^2\sqrt{1-2M/r}\,\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta\Phi) = 4\pi\rho. \tag{4}$$

(e) Now let us perform a Legendre polynomial expansion, in which

$$\rho(r,\theta) = \sum_{\ell=0}^{\infty}\rho_\ell(r)P_\ell(\cos\theta), \tag{5}$$

and similarly for Φ. Derive a 2nd order ODE (in r) for Φ_ℓ. Convert this equation to $r_\star$ to yield

$$-\partial_{r_\star}\left(\frac{r^2}{1-2M/r}\partial_{r_\star}\Phi_\ell\right) + \ell(\ell+1)\Phi_\ell = 4\pi r^2\sqrt{1-2\frac{M}{r}}\,\rho_\ell. \tag{6}$$

3. Electrostatics near a black hole – Part II. [14 points]
This is a continuation of Problem #2, but from the solution above to 2(d) the rest can be worked independently.

Integration of the $\ell = 0$ equation shows that

$$-\frac{r^2}{1-2M/r}\partial_{r_\star}\Phi_0 = \int 4\pi r^2\sqrt{1-2\frac{M}{r}}\,\rho_0(r)\,dr_\star. \tag{7}$$

The left-hand side taken as $r_\star \to \infty$ is the net charge (it is r^2 times the electric field), so we interpret the definite integral of the right-hand side from $2M$ to ∞ to be the charge outside the hole. The integration constant (value of the left-hand side at $2M$) is interpreted as the charge of the hole itself.

In this problem, assume the black hole itself is uncharged and that the external charge is a point above the North pole of charge q at some radius R.

(a) Show that the expression for $\rho_\ell(r)$ is:

$$4\pi r^2\sqrt{1-2\frac{M}{r}}\,\rho_\ell(r) = (2\ell+1)q\delta(r_\star - R_\star). \tag{8}$$

(b) At $r < R$ or $r > R$, Φ_ℓ obeys a homogeneous equation. Find the solutions to this equation in the limits [i] far from the hole, $r_\star \gg M$; and [ii] near the horizon, $-r_\star \gg M$.

(c) Show that the condition of bounded electric fields as one approaches the hole and zero potential at ∞ allows only one solution in each regime, [i] and [ii].

(d) Consider the regime of $R - 2M \ll 2M$, i.e. the charge is very close to the hole. Using the theory of ODEs and the homogeneous solutions, show that as this limit is taken, the electric potential seen outside the hole becomes spherically symmetric – that is, the apparent electric dipole moment, quadrupole moment, etc. measured by observers far from the hole and stationary charge decreases. How do they scale with $R - 2M$?

(e) Do a qualitative comparison of (d) to what happens for a point charge very close to the surface of a neutral conducting sphere (you may look the answer to this up in an electrostatics text, or draw a picture; there is no need for extensive new calculations for this part).