Ph 236 – Homework 12

Due: Friday, February 17, 2012

1. External effect of radial perturbations. [10 points]
(a) Using the equations for radial perturbations of spherically symmetric stars in the notes (or in MTW Ch. 26), show that exterior to a star $\delta\Lambda(t,r) = 0$ and that $\delta\Phi(t,r)$ is only a function of t: $\delta\Phi(t,r) = \delta\Phi(t)$.

(b) Show that there is a residual freedom in the choice of coordinates that allows us to set $\delta\Phi(t,r) = 0$ in the exterior. Thus the external geometry is still Schwarzschild.

This problem has two major implications, at least in perturbation theory (although both are valid in general) – (i) that, just like in Newtonian gravity, spherically symmetric pulsations, even if dynamic, have no effect on the exterior field; and (ii) the radial pulsations of a spherical star, no matter how massive, emit no gravitaitonal waves.

2. The external Universe seen by a hovering observer. [14 points]
Suppose that you are hovering in a rocketship near a black hole at Schwarzschild coordinate r, with $r \le 3M$, looking up away from the black hole at the "sky."

(a) A photon falls in from afar, with initial energy $h\nu_\infty$, and is seen by the observer with frequency $h\nu_{\rm obs}$. As a function of r, what is $\nu_{\rm obs}/\nu_\infty$?

(b) Suppose that an observer looks for photons coming in at an angle ζ from the vertical. We are interested in whether the observer sees light from ∞. Without loss of generality, place the observer on the equator and the photon's trajectory in the equatorial plane; show that the ratio of the photon's energy to angular momentum is

$$\frac{p_\phi}{-p_t} = \frac{r\sin\zeta}{\sqrt{1-2M/r}}. \tag{1}$$

Using conservation laws, and the condition that the photon must have come in from infinity along a trajectory with $(p_r)^2 \ge 0$, find the maximum value ζ_0 of ζ for which the observer sees the "sky" (external Universe). At larger values of ζ the observer must see the black hole (or, as stated in class, the highly redshifted surface of the star that formed it – more on this next week). Make a graph of your answer.

(c) Explain qualitatively why $\zeta_0 \to \pi/2$ at $r = 3M$, and $\zeta_0 \to 0^+$ at $r = 2M$. [I'm looking for a couple sentences, not an involved calculation.]

(d) How many images of each distant star are seen by the observer? [Again a few sentences and a picture suffice – you should not need a long calculation for this.]

(e) The real external Universe is filled with the cosmic microwave background. At what radius coordinate r does the hovering observer see the CMB in the optical (say, as a blackbody with the temperature of the Sun)? What is the apparent angular diameter of the disk formed by the CMB?

3. The external Universe seen by an infalling observer. [12 points]
Now let us consider Problem #2, but we will have the observer fall inward radially from infinity with unit specific energy ($u_t = -1$). Assume in this problem that the infalling observer is at $2M < r \le 3M$.

(a) When the infalling observer $\mathcal{O}$ reaches r, what is their 3-velocity v with respect to the stationary observer? [Hint: you can use $\mathcal{O}$'s 4-velocity from class.]

(b) Just as the hovering observer sees the Universe as a disk of light above them, so the infalling observer $\mathcal{O}$ sees the Universe as a disk of light behind them. Using the special relativistic formula for aberration of light (which you need not re-derive), find the angular radius Ξ of this disk as seen by the infalling observer.

(c) Graph your answer to (b). Show that as $r \to 2M$, Ξ approaches a constant. What is its numerical value? [This implies that at horizon crossing, nothing special happens to $\mathcal{O}$'s ability to *see* the outside world; all that changes is that $\mathcal{O}$ can no longer send messages back out of the hole.]

(d) What is the ratio of observed to "initial" frequency $\nu_{\rm obs}/\nu_\infty$ for a photon at the center of the disk (i.e. falling radially inward and catching up with the observer)? Graph this quantity and show that – unlike the case for the stationary observer – it remains finite even as the observer reaches $r = 2M$.