Ph 236 – Homework 10
Due: Friday, January 27, 2012

1. Thermodynamic relations obtained via conservation laws. [6 points]
Suppose that the baryon number density n and pressure p of a perfect fluid are just functions of the density ρ. Suppose also that the baryon current $I^\mu = nu^\mu$ is conserved, $\nabla \cdot \boldsymbol{I} = 0$. Prove that, as claimed in class,

$$\frac{d(\rho n^{-1})}{dn} = \frac{p}{n^2}. \tag{1}$$

2. Appearance of a spherical star. [14 points]
Consider a spherical star of mass M and radius R. We will now consider the motion of a photon outside the star, assuming that it is in the equatorial plane.

(a) Consider a photon approaching (or receding from) the star at impact parameter b, with energy ϵ as measured by an observer at rest relative to the star at large radius. As a function of r, write down the covariant and contravariant components of the momentum.

(b) What is the legal range of radii that the photon can traverse?

(c) Determine the maximum impact parameter at which the photon impacts the star (or must have come from the star) instead of simply flying past. Explain why this value is the apparent radius $R_{\rm app}$ of the star seen by an external observer.

(d) Your answer to part (c) should depend on whether R is greater than, less than, or equal to $3M$. What is the physical interpretation of this result?

(e) Your $R_{\rm app}$ is different from the radius R determined by setting the star's surface area equal to $4\pi R^2$. What is the difference $R_{\rm app}/R - 1$ for [i] the Sun; [ii] a typical white dwarf; and [iii] a typical neutron star?

3. Charged spherical objects. [16 points]
We do not expect this situation astrophysically, but it is instructive to consider a charged spherical system with an electric field. Let's consider here the exterior (vacuum) solution. The metric is

$$ds^2 = -e^{2\Phi}\, dt^2 + e^{2\Lambda}\, dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{2}$$

where Φ and Λ are functions of r.

(a) Show that symmetry considerations require the electromagnetic field tensor to take the form:

$$F_{\hat r\hat t} = -F_{\hat t\hat r} = E, \tag{3}$$

with all other components zero, where E is the radial electric field, in the orthonormal basis defined by Notes XVIII Eq. (15).

(b) Using Maxwell's equations, find the dependence of E on radius r. Show that the limiting behavior at large radii is $E \approx Q/r^2$, where Q is a constant. Explain the meaning of Q.

(c) What is the stress-energy tensor $T_{\hat\mu\hat\nu}$ associated with the electromagnetic field (in the orthonormal basis)?

(d) What is $R_{\hat\mu\hat\nu}$ as implied by Einstein's equations?

(e) Show that Φ and $-\Lambda$ differ by a constant; and that this constant can be eliminated by a coordinate transformation so we may take $\Phi = -\Lambda$. [*Hint*: this is not mathematically tedious; you just need to find the right combination of Einstein's equations.]

(f) Using another component of Einstein's equations, derive a first-order ODE for $\Lambda(r)$. Solve this equation to derive the full exterior metric structure and electromagnetic field. You should need to introduce a second parameter, in addition to Q; choose this to be the total mass M.

The solution of this problem also describes the exterior (and, with appropriate analytic continuation, the interior) of a charged spherical black hole.