Ph 236 – Homework 1
Due: Friday, October 7, 2011

1. Index exercises in special relativity. [12 points]

(a) MTW Exercise 2.2.
(b) MTW Exercise 2.4.

2. Lorentz transformations. [12 points]
Consider the standard Lorentz transformation associated with a boost along the 1-axis, i.e.

$$\mathbf{L} = \begin{pmatrix} \frac{1}{\sqrt{1-\beta^2}} & -\frac{\beta}{\sqrt{1-\beta^2}} & 0 & 0 \\ -\frac{\beta}{\sqrt{1-\beta^2}} & \frac{1}{\sqrt{1-\beta^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1}$$

with $-1 < \beta < 1$.

(a) Prove directly that the special relativity metric tensor $\eta_{\mu\nu}$ is invariant, i.e. $g_{\mu'\nu'}$ is described by the same matrix diag$(-1, 1, 1, 1)$.

(b) Consider a particle whose 3-velocity in the unprimed frame is V, directed along the 1-axis. What is its 4-velocity u^μ expressed in the unprimed coordinate system?

(c) Transform the 4-velocity to the primed system. What is the 3-velocity seen by the primed observer?

3. Normal forms of the metric tensor. [12 points]
This problem will justify the claims made in lecture (Lecture Notes I, §IIIB) about the possible forms of the symmetric tensor $g_{\mu\nu}$. Assume an n-dimensional space(time). In this problem any real basis transformation with invertible transformation matrix $\mathbf{L} \in$GL$(n, \mathbb{R})$ is legal; complex transformations are disallowed.

(a) Prove that there exists a basis transformation that diagonalizes $g_{\mu\nu}$.

(b) Prove that a further basis transformation can set all diagonal entries equal to -1, 0, or $+1$ (while leaving $g_{\mu\nu}$ diagonal). In such a case, each basis vector can be categorized as either having negative, zero, or positive square-norm.

Let us now call the number of -1, 0, and $+1$ diagonal entries n_-, n_0, and n_+ (with $n_- + n_0 + n_+ = n$). We will now prove that the signature (n_-, n_0, n_+) is unique. Suppose that there were two different choices of basis (unprimed and primed) satisfying part (b), *but* with a different signature $(n_-, n_0, n_+) \neq (n'_-, n'_0, n'_+)$. Call the basis transformation matrix $\mathbf{L}$.

(c) Suppose that we had $n_- > n'_-$. Then prove that this implies that there is a nonzero vector $\boldsymbol{v}$ that satisfies the following conditions: [I] $v^{\alpha'} = 0$ for α' corresponding to any primed basis vector of negative square norm; and [II] $v^\beta = 0$ for β corresponding to any unprimed basis vector of zero or positive square norm. [*Hint*: consider the number of conditions imposed on $\boldsymbol{v}$ and the number of degrees of freedom.] Prove that calculations in the unprimed and primed bases lead to contradictory conclusions regarding the sign of $\boldsymbol{v} \cdot \boldsymbol{v}$, and hence that our assumption was impossible.

(d) Explain why arguments similar to those in (c) tell us that we must have $n'_- = n_-$, $n'_+ = n_+$. Then prove that $n'_0 = n_0$.