Ph 236 – Final Exam, Winter Term
Due: Wednesday, March 21, 2012

1. Orbits falling into Kerr holes, and the thermodynamic consequences. [10 points]
Consider a small particle that falls into a Kerr black hole.

(a) Show that in order for the particle to reach the future horizon, the energy per unit mass $\mathcal{E}$ and angular momentum per unit mass $\mathcal{L}$ satisfy the relation

$$\mathcal{E} > \Omega_{\rm H}\mathcal{L}. \tag{1}$$

(b) Consider the horizon area S. Show that

$$\left.\frac{\partial S}{\partial J}\right|_M = -\Omega_{\rm H}\left.\frac{\partial S}{\partial M}\right|_J. \tag{2}$$

Explain why this, combined with the result from part (a), shows that the hole's surface area can only grow if small particles are dumped in.

(c) In thermodynamics, recall that any conserved extensive quantity q is conjugate to a conserved intensive quantity p via

$$p = T\left.\frac{\partial S}{\partial q}\right|_E, \tag{3}$$

where E is the total energy (and other extensive quantities remain fixed) and T is the temperature; this intensive quantity is equilibrated between subsystems in thermodynamic equilibrium.[1] What intensive quantity is conjugate to the conserved total angular momentum? Does it make sense to say that this quantity is "constant" on the horizon (independent of θ and ϕ) and if so is this in fact the case?

(d) Suppose you look at a Kerr black hole down the North Pole. Do you expect the Hawking radiation to be polarized, and if so in what sense?

2. Innermost stable circular orbits. [10 points]
This problem investigates some properties of the circular equatorial orbits around the Kerr spacetime.

Solve the problem in the limits: [i] $\chi = -1$; [ii] $\chi = 0$; and [iii] $\chi = 1 - \epsilon$ (to leading order in ϵ).

(a) The energy $\mathcal{E}$ and angular momentum $\mathcal{L}$ of the innermost stable circular orbit.

(b) The orbital (angular) frequency Ω_ϕ at the innermost stable circular orbit. For case [iii] compare to $\Omega_{\rm H}$. (Hint: $d\phi/dt$ is constant so you don't need the funny averages.)

(c) Numerically evaluate (b) for a $20M_\odot$ black hole for each case. Which case would lead to a gravitational wave signal that extends to the highest frequencies? (I don't expect detailed calculations for this part – just a qualitative argument is sufficient.)

[1] For example, if total volume is conserved then it is conjugate to pressure.