Ph 236 – Final Exam, Fall Term
Due: Friday, December 9, 2011

1. Conservation laws. [10 points]
(a) Consider a particle following along a geodesic with some 4-velocity $u^\alpha = dx^\alpha/d\lambda$. Suppose that ξ^α is a vector field satisfying

$$\xi_{(\alpha;\beta)} = 0. \tag{1}$$

Show that the inner product $K = u^\alpha \xi_\alpha$ is conserved along the trajectory.

(b) A vector field satisfying Eq. (1) is called a *Killing field*. Show that if ξ^α and ψ^α are Killing fields, then also the commutator

$$\chi^\alpha = \xi^\gamma \psi^\alpha{}_{;\gamma} - \psi^\gamma \xi^\alpha{}_{;\gamma} \tag{2}$$

is a Killing field. [*Hint*: Explicitly expand the covariant derivative $\chi_{(\alpha;\beta)}$, and use the Riemann tensor.]

2. Cosmic strings – Part I. [10 points]
Until the mid-1990s, a popular model for the formation of structure in the Universe was the *cosmic string*.[1] This is a 1-dimensional object predicted by many theories beyond the Standard Model with some energy per unit length μ. The tension is also equal to μ. This problem will use **linearized** GR to investigate the properties of cosmic strings.

(a) Consider a cosmic string aligned in the 3-direction, with stress-energy tensor

$$T^{00} = -T^{33} = \mu\delta(x^1)\delta(x^2) \tag{3}$$

and other components zero. Show that $T^{\mu\nu}{}_{,\nu} = 0$.

(b) Using linear theory in the Lorentz gauge, find the differential equation satisfied by $\bar{h}_{\mu\nu}$.

(c) Show that if we define $\varpi \equiv \sqrt{(x^1)^2 + (x^2)^2}$, then

$$\nabla^2 \ln \frac{\varpi}{L} = 2\pi\delta(x^1)\delta(x^2), \tag{4}$$

where L is any constant. Use this to find the cylindrically symmetric solutions for $\bar{h}_{\mu\nu}$. These may contain an arbitrary length L. [We will return to the physical significance of L in Problem #3(b).]

(d) Find $h_{\mu\nu}$. Show that all components other than h_{11} and h_{22} are zero, and that for some constant C:

$$h_{11} = h_{22} = 2C \ln \frac{\varpi}{L}. \tag{5}$$

What is C in terms of μ?

3. Cosmic strings – Part II. [10 points]
This is really a continuation of Problem #2, but will be scored independently. Starting from the answer to #2(d), we see that the metric describing a cosmic string is

$$ds^2 = -dt^2 + \left(1 + 2C \ln \frac{\varpi}{L}\right) [(dx^1)^2 + (dx^2)^2] + (dx^3)^2, \tag{6}$$

where $|C| \ll 1$ and L is some arbitrary length and $\varpi \equiv \sqrt{(x^1)^2 + (x^2)^2}$. You may work this problem to first order in C.

(a) Transformation the coordinates to t, $z = x^3$, ϖ and $\phi = \arctan(x^2/x^1)$. Show that the result is

$$ds^2 = -dt^2 + \left(1 + 2C \ln \frac{\varpi}{L}\right) d\varpi^2 + \left(1 + 2C \ln \frac{\varpi}{L}\right) \varpi^2\, d\phi^2 + dz^2. \tag{7}$$

[1] We're still looking for these, but none have been found.

(b) Now transform the radial coordinate ϖ into another coordinate σ chosen to make the angular term look simple, i.e. put it in the form

$$ds^2 = -dt^2 + f(\sigma)\, d\sigma^2 + (1+2C)\sigma^2\, d\phi^2 + dz^2. \tag{8}$$

(The $1+2C$ is for later convenience.) What is $f(\sigma)$? Show that it does not depend on L, and explain what this says about the physical significance (or not) of the parameter L.

(c) Show that under the transformation $\phi' = (1+C)\phi$ that the answer to (b) transforms into the cylindrical coordinate description of Minkowski spacetime, but with a funny feature: you return to your original starting point after increasing the longitude by $\Delta\phi' = 2\pi(1+C)$ instead of 2π. This is known as a "conical singularity" or "angle-defect singularity."

(d) Explain why an observer near the cosmic string cannot infer its presence gravitationally unless the experimental setup either contains the cosmic string or contains a loop enclosing it. Suggest a method by which one would be able to search for cosmic strings through astronomical observations. [*Note*: for a stable string with positive tension, $C < 0$.]