1 Galaxy surveys and large scale structure

The CMB provides a wealth of data at high redshift, but if we want to understand the more recent evolution of the Universe we must observe nearby objects. The easiest of these to observe are galaxies. Most of the cosmological information from galaxies comes from their distribution, since this is easiest to predict from first principles. We will first discuss some general properties of galaxies, and then consider methods of extracting cosmological information:

- Angular correlations.
- Redshift surveys.
- ISW effect.
- Bispectrum/non-Gaussianity.
- Weak lensing.

We do not have a fundamental theory of galaxy formation, and will not for some time to come. However the processes of gravitational collapse, gas accretion, star formation, etc. are local and smoothed on some scale R the number density of galaxies should depend only on the local density of the matter (and possible stochastic processes). That is:

$$n_g(\mathbf{x}) = n_g[\delta(\mathbf{x})] + \text{stochastic}. \tag{1}$$

The longest-range effect in galaxy formation (with the possible exception of reionization) is gravitational collapse, which acts on a scale $R \sim k_\star^{-1}$ where $\Delta_\delta^2(k_\star) \sim 1$. (In order to enhance the density by of order unity on scale R, the matter must move a distance $\sim R$ and the use of the linear density field is inappropriate on smaller scales.) Smoothed on larger scales we will have $|\delta| \ll 1$, and can make a Taylor expansion:

$$n_g(\mathbf{x}) = \bar{n}_g[1 + b\delta(\mathbf{x})] + \text{stochastic}, \tag{2}$$

where b is a number called the bias. Both n_{g0} and b may depend on the type of galaxy and the redshift. Some typical examples are:

- "Average" galaxies (luminosity $\sim$ Milky Way) at $z = 0$: $n_g \sim 10^{-2}h^3$ Mpc^{-3}; $b \sim 1$.
- Luminous red galaxies (LRGs – large ellipticals) at $z \sim 0.4$: $n_g \sim 4 \times 10^{-4}h^3$ Mpc^{-3}; $b \sim 2$.
- Type 1 quasars (unobscured), $z \sim 1.5$: $n_g \sim 10^{-5}h^3$ Mpc^{-3}; $b \sim 2.5$. Number density declines at high redshift but bias increases: at $z > 3$ the bias increases to $b \sim 10$.

At the present most galaxy clustering constraints come from these types of objects. Typical galaxies are numerous but not very bright ($\sim 10^{10}L_\odot$) and have only been mapped over large fractions of the sky out to $z \sim 0.2$ by 2dF and SDSS (optical) and 2MASS (infrared - photometric). LRGs are brighter ($\sim 10^{11}L_\odot$) and have been mapped to $z \sim 0.4$ by SDSS (and out to ~ 0.6 without spectra). Quasars can exceed $10^{12}L_\odot$ and emit strongly in the UV so that they are visible at $z > 1$; they also have strong emission lines which helps in measuring the redshift. In the future, the more numerous star-forming galaxies at $0.5 < z < 2$ may overtake quasars as the best tool for large scale structure, due to improvements in IR observations.

2 Angular correlations

The angular distribution of galaxies are the simplest type of galaxy clustering measurement to make. The method does not require a redshift for each individual galaxy, rather it only requires the redshift distribution; this is an advantage because obtaining a fair sample of redshifts is easier than measuring 10^6 redshifts.

Let us define the galaxy overdensity on the sky g as the number of galaxies in a pixel divided by the mean number, minus 1:

$$g = \frac{N}{\bar{N}} - 1. \tag{3}$$

The number of galaxies N can be obtained by an integral over the line of sight,

$$N = \Omega \int n_g r^2 \, dr, \tag{4}$$

where Ω is the solid angle of the pixel. Then:

$$\begin{aligned} g &= \frac{\int n_g r^2 \, dr}{\int \bar{n}_g r^2 \, dr} - 1 \\ &= \frac{\int \bar{n}_g [1 + b\delta] r^2 \, dr}{\int \bar{n}_g r^2 \, dr} - 1 \\ &= \frac{\int \bar{n}_g b \delta r^2 \, dr}{\int \bar{n}_g r^2 \, dr}. \end{aligned} \tag{5}$$

The density field δ is actually changing with redshift according to the growth function:

$$\delta(\mathbf{x}, z) = \frac{D(z)}{D(0)} \delta_0(\mathbf{x}). \tag{6}$$

It is common to define the window function:

$$f(r) = \frac{D(z)}{D(0)} \frac{b(r)\bar{n}_g(r)}{\int \bar{n}_g(r) \, r^2 \, dr}. \tag{7}$$

If the bias is constant then the selection function integrates to the bias: $\int r^2 f(r)\, dr = b$. In terms of this, the galaxy overdensity is:

$$g(\hat{\mathbf{n}}) = \int r^2 f(r) \delta_0(r\hat{\mathbf{n}})\, dr. \tag{8}$$

Just like the case of the CMB, we cannot predict the specific distribution g that we will see; rather we predict statistical properties. These are the angular power spectrum and the angular correlation function. The angular power spectrum is obtained by taking the spherical harmonic transform of g:

$$g(\hat{\mathbf{n}}) = \sum_{lm} g_{lm} Y_{lm}(\hat{\mathbf{n}}), \tag{9}$$

with variance:

$$\langle g^*_{lm} g_{l'm'} \rangle = C_l^{gg} \delta_{ll'} \delta_{mm'}. \tag{10}$$

The statistics of g_{lm} can be obtained as follows. First we consider the $m = 0$ case, and write it in terms of δ_0:

$$g_{l0} = \int f(r) \delta_0(r\hat{\mathbf{n}}) Y^*_{l0}(\mathbf{n})\, r^2\, dr\, d^2\hat{\mathbf{n}}. \tag{11}$$

Letting $\mathbf{x} = r\hat{\mathbf{n}}$, and using the Fourier expansion of the density field:

$$\delta_0(\mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \delta_0(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{12}$$

and so:

$$g_{l0} = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \delta_0(\mathbf{k}) \int f(r) e^{i\mathbf{k}\cdot\mathbf{x}} Y^*_{l0}(\mathbf{n})\, r^2\, dr\, d^2\hat{\mathbf{n}}. \tag{13}$$

The angular integration gives:

$$\int e^{i\mathbf{k}\cdot\mathbf{x}} Y^*_{l0}(\mathbf{n})\, d^2\hat{\mathbf{n}} = i^l \sqrt{4\pi(2l+1)}\, j_l(kr) P_l(\hat{k}_3), \tag{14}$$

so

$$g_{l0} = i^l \sqrt{4\pi(2l+1)} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \delta_0(\mathbf{k}) \int f(r)\, r^2 j_l(kr)\, dr. \tag{15}$$

The squared absolute value gives:

$$C_l^{gg} = \langle |g^2_{l0}| \rangle = 4\pi(2l+1) \int \frac{d^3\mathbf{k}}{(2\pi)^3} P_\delta(k) \left| \int f(r)\, r^2 j_l(kr)\, dr \right|^2 P_l^2(\hat{k}_3). \tag{16}$$

The angular average of $P_l^2(\hat{k}_3)$ is:

$$\frac{1}{4\pi} \int d^2\hat{\mathbf{k}} P_l^2(\hat{k}_3) = \frac{1}{2} \int_{-1}^{1} d\hat{k}_3\, P_l^2(\hat{k}_3) = \frac{1}{2l+1}, \tag{17}$$

so

$$C_l^{gg} = 4\pi \int d\ln k\ \Delta_\delta^2(k) \left| \int f(r)\, r^2 j_l(kr)\, dr \right|^2. \tag{18}$$

This equation ignores the stochastic part, which is usually modeled as Poisson noise. This adds $1/n_{2D}$ to the power spectrum, where n_{2D} is the number of galaxies per steradian.

In the limit of large l, if $f(r)$ is slowly varying, one may simplify this equation further. The function $j_l(x)$ is near zero for $x < l$, and oscillates for $x > l$. Neither regime contributes to the integral. In between there is a bump in $j_l(x)$, which may be modeled by replacing with a δ-function:

$$j_l(x) \to \sqrt{\frac{\pi}{2l}}\delta(x-l). \tag{19}$$

This simplifies the integral to:

$$C_l^{gg} = \frac{2\pi^2}{l} \int d\ln k\ \Delta_\delta^2(k)[f(r)]^2 \frac{r^4}{k^2}, \tag{20}$$

where $r = l/k$. We may then change the integration variable from k to r:

$$C_l^{gg} = \frac{2\pi^2}{l} \int d\ln r\ \Delta_\delta^2 \left(\frac{l}{r}\right) [f(r)]^2 \frac{r^6}{l^2}, \tag{21}$$

or after simplifying:

$$C_l^{gg} = \frac{2\pi^2}{l^3} \int dr\ r^5 \Delta_\delta^2 \left(\frac{l}{r}\right) [f(r)]^2. \tag{22}$$

This is usually written in terms of $P(k)$:

$$C_l^{gg} = \int \frac{dr}{r^2}\ P_\delta \left(\frac{l}{r}\right) [r^2 f(r)]^2. \tag{23}$$

A related statistic is the angular correlation function:

$$\langle g(\hat{\mathbf{n}}_1) g(\hat{\mathbf{n}}_2) \rangle = \xi(\theta), \tag{24}$$

where θ is the angle between $\hat{\mathbf{n}}_1$ and $\hat{\mathbf{n}}_2$. One can show (exercise!) that:

$$\xi(\theta) = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} C_l^{gg} P_l(\cos\theta). \tag{25}$$

Angular clustering can be done with very large galaxy samples over large volumes. Nevertheless it has disadvantages. One is that the third dimension is lost: projection destroys many of the Fourier modes that could have been measured in a redshift survey. As we will see these modes are not similar to the transverse modes because the peculiar velocities of galaxies modify their observed redshifts in a way that can be used to extract information. Finally, the sample of galaxies under consideration will be inhomogeneous: the nearby galaxies will be fainter than average, and the more distant galaxies will be brighter because those are the only ones we can see. These types of galaxies have different biases, which have caused trouble for many cosmologists. One way to get around this is to use color selection of galaxies to pick out a particular redshift range, as has been done very successfully with LRGs.

3 Redshift surveys

The galaxy power spectrum can also be measured in 3-dimensional surveys if the redshift of a galaxy is measured as well as its position. Redshifts are usually obtained with multi-object spectrographs where fibers or slits are placed on objects of interest, then run through a grism or diffraction grating to spread the light into colors, which are imaged onto a CCD. In this way many galaxy spectra are obtained at once. The choice of spectral features to determine the redshift varies from object to object:

- For old galaxies (e.g. LRGs) the most readily identifiable features are metal absorption lines, e.g. Ca II, Mg II, and Na I.
- For star-forming galaxies one can often find emission features: the forbidden lines (O II, O III) and Balmer lines (Hα, Hβ).
- Quasars have a host of emission lines; in order of wavelength: Lyα, Si IV, C IV, C III, Mg II, Hγ, Hβ, O III, Hα. The O III lines and Balmer lines are good choices because they trace the rest frame wavelength of the quasar. Other lines such as Lyα, C IV, Mg II are complex and asymmetric and may be significantly shifted by scattering of the blue side of the line by outflows or IGM. They are less desirable but may be the best choice at high redshift.

There are three major features that have historically been used in redshift surveys to probe cosmology.

Broadband shape. On linear scales the galaxy power spectrum is proportional to the matter power spectrum, $\Delta_g^2(k) = b^2\Delta_m^2(k)$. We know that these power spectra are $\propto k^{3+n_s}$ on large scales but roll over to $k^{3+n_s}T(k)^2$ on small scales, with the rollover point occuring at $k = k_{eq} \propto \Omega_m h^2$. One of the major goals of the last generation of redshift surveys (Sloan Digital Sky Survey/SDSS; 2 degree Field/2dF) was to precisely measure the rollover scale.

However one should remember that in a redshift survey one measures z, not r. The distance is inferred from Hubble's law: if $z \ll 1$ then $r = z/H_0$. Since we don't know H_0 a priori, r is measured in units of $h^{-1}\,\mathrm{Mpc}$, not Mpc. Thus when we take the 3D power spectrum, the wavenumber k and hence k_{eq} are measured in units of $h\,\mathrm{Mpc}^{-1}$. Therefore what the shape of the galaxy power spectrum can measure is:

$$\frac{k_{eq}}{h} \propto \frac{\Omega_m h^2}{h} = \Omega_m h. \tag{26}$$

This product was often called the *shape parameter* Γ, although in recent years this term has fallen out of favor. In combination with $\Omega_m h^2$ from the CMB, Γ allows one to separately measure Ω_m and H_0. Alternatively with a measurement of H_0, Γ allows one to measure Ω_m. This latter combination was one of the early pieces of evidence that $\Omega_m < 1$.

The advantage of Γ is that it can be measured with a small sample of galaxies (10^4). However one must worry about nonlinear evolution in the power spectrum or interactions between galaxies that cause small amounts of k-dependent bias. Simulations typically find $k < 0.1h\,\mathrm{Mpc}^{-1}$ to be linear scales, but if one probes to the percent level this may not be a safe assumption. Therefore in the future cosmologists will turn to another way of using galaxy clustering.

Baryon oscillations. The wiggles in the power spectrum due to baryonic effects in $T(k)$ are much more robust than the broadband shape: to move them requires a nonlocal interaction at the 150 Mpc scale. Therefore measuring these features (whose physical size is known from CMB) is a promising way to constrain cosmology. In fact one can measure r from the transverse scale and H from the radial scale, so the baryon oscillation is very rich in information.

Redshift-space distortions. So far we've assumed that the redshift z is simply related to the distance r to a galaxy using the usual FRW equations, but that's not quite true. Galaxies have peculiar velocities and these are correlated with the distribution of matter. This is not a small effect, even though the peculiar velocities are $\ll c$: they produce order-unity effects on the power spectrum measured in a redshift survey. To see this, we note that the observed redshift is related to the true redshift via:

$$1 + z_{\mathrm{obs}} = \frac{\lambda_{\mathrm{obs}}}{\lambda_{\mathrm{gal}}} = \frac{\lambda_{\mathrm{obs}}}{\lambda_{\mathrm{com}}}\frac{\lambda_{\mathrm{com}}}{\lambda_{\mathrm{gal}}} = (1 + z_{\mathrm{true}})(1 + v_\parallel), \tag{27}$$

where $v_\parallel$ is the line-of-sight velocity. For small velocities, the distance at which we place the galaxy in a 3D map is then:

$$r_{\mathrm{obs}} = r_{\mathrm{true}} + \frac{1+z}{H} v_\parallel. \tag{28}$$

To see the quantitative effect of the galaxy velocity, let's go to linear perturbation theory. The parallel component of the velocity is $v_\parallel = \mu v$, where μ is the cosine of the angle between the line of sight $\hat{\mathbf{n}}$ and wavevector $\hat{\mathbf{k}}$: $\mu = \hat{\mathbf{n}} \cdot \hat{\mathbf{k}}$. In most cases, the galaxies will move at the same bulk velocity as the dark matter, due to the equivalence principle. (The exception is satellite galaxies; see below.) Then the velocity is given by the continuity equation:

$$v = i\frac{\dot{\delta}}{k}, \tag{29}$$

so

$$r_{\rm obs} = r_{\rm true} + i\frac{\mu}{aHk}\dot{\delta}. \tag{30}$$

In linear perturbation theory, the 3D galaxy overdensity $\delta_g(\mathbf{k})$ in a given Fourier mode can then be determined. It is the physical overdensity $b\delta$ corrected for the Jacobian of the transformation from true coordinates (or "real space") to observed coordinates (or "redshift space"):

$$1 + \delta_g(\mathbf{x}) = [1 + b\delta(\mathbf{x})] \left(\frac{dr_{\rm obs}}{dr_{\rm true}}\right)^{-1} = 1 + b\delta(\mathbf{x}) - \frac{d}{dr}\left(\frac{\mu}{aHk}\dot{\delta}\right), \tag{31}$$

where we have worked only to first order. Making the replacement $d/dr \to ik_\parallel = ik\mu$, we get:

$$\delta_g = b\delta + \frac{\mu^2}{aH}\dot{\delta}. \tag{32}$$

Now in linear perturbation theory, $\dot{\delta} = (\dot{D}/D)\delta$, where D is the growth function. Thus:

$$\delta_g = b\left(1 + \frac{\dot{D}}{aHDb}\mu^2\right)\delta. \tag{33}$$

We define the redshift-space distortion parameter:

$$\beta = \frac{\dot{D}}{aHDb}. \tag{34}$$

Then the power spectrum of galaxies in redshift space is:

$$\Delta^2_g(\mathbf{k}) = b^2\left(1 + \beta\mu^2\right)\Delta^2_\delta(k). \tag{35}$$

The fact that the power spectrum depends on the direction of $\mathbf{k}$ (relative to the line-of-sight) in addition to its magnitude opens up a new tool for cosmology. In an Einstein-de Sitter universe, we would have $D \propto a$ and then $\dot{D}/aHD = 1$. In ΛCDM models this ratio is closer to $\dot{D}/aHD \approx \Omega_m^{0.6}$. But if Ω_m is known, e.g. from the shape parameter, and we measure β, this means we can obtain the bias and convert the observed galaxy power spectrum into the true matter power spectrum.

A complicating fact is that in redshift space there are nonlinear corrections that are significant even at $k < 0.1h\,{\rm Mpc}^{-1}$. Chief among these are the so-called *fingers of God*: satellite galaxies in groups and clusters that have very large orbital velocities, and hence in redshift space form an elongated feature pointed at the observer. Observers interested in β usually compress these fingers of God by identifying the central galaxy (at least statistically) and collapsing the entire cluster to its velocity. An alternative is to remove the satellites from the analysis entirely. All of these effects are helped by going to the largest scales possible, and future large-volume redshift surveys are promising ways to measure β.

Normalization. Historically the galaxy power spectrum was much easier to measure than the bias, and hence one of the cosmological parameters over which there was considerable debate was the normalization of the matter power spectrum. The CMB observers would usually measure the normalization by quoting $\Delta_\zeta(k_\star)$ at some scale $k_\star$. Low-redshift observers prefer a different normalization convention, preferring to quote σ_8: the linear-theory standard deviation of δ_8, the matter overdensity in a sphere of radius $R = 8h^{-1}\,\mathrm{Mpc}$. One can find the variance of the filtered density field δ_8 via:

$$\sigma_8^2 = \langle \delta_8^2 \rangle = \int \frac{dk}{k} \Delta^2_{\delta_8}(k) = \int \frac{dk}{k} |W(k)|^2 \Delta^2_\delta(k), \tag{36}$$

where $W(k)$ is the Fourier transform of a top-hat function with radius R:

$$W(k) = \left(\frac{4}{3}\pi R^3\right)^{-1} \int_V e^{i\mathbf{k}\cdot\mathbf{x}}\, d^3\mathbf{x}. \tag{37}$$

The conversion from $\Delta_\zeta(k_\star)$ to σ_8 depends on the other cosmological parameters; standard packages like CMBFAST and CAMB will do the conversion. The WMAP estimate is $\sigma_8 = 0.80 \pm 0.04$, although most low-redshift measurements prefer a slightly larger value.

4 ISW effect

Most of the CMB anisotropy is formed at very high redshift where there are no galaxies. However the ISW effect, which involves changes in the gravitational potential giving energy to (or taking energy from) the CMB photons, is active at late times. The temperature perturbation due to ISW is:

$$\Theta(\hat{\mathbf{n}})|_{\mathrm{ISW}} = 2 \int \dot{\Phi}(r\hat{\mathbf{n}})\, dr. \tag{38}$$

This is zero in the matter-dominated era because Φ =constant. However the Poisson equation gives:

$$\Phi = -\frac{4\pi G a^2}{k^2} \bar{\rho}_m \delta, \tag{39}$$

and since $\bar{\rho}_m \propto a^{-3}$ and $\delta \propto D(a)$, the potential varies with time in proportion to $D(a)/a$. Thus in a ΛCDM universe, where D grows more slowly than a, the potentials move toward zero. In overdense regions $\dot{\Phi} > 0$, and in underdense regions $\dot{\Phi} < 0$. This leads to a small additional anisotropy in the CMB, but more importantly it leads to a positive correlation of the CMB with the galaxy distribution. The effect is unique to models with Λ or curvature and its detection rules out Einstein-de Sitter.

This galaxy-CMB cross-correlation was first observed by combining the WMAP CMB data with several galaxy surveys (2MASS, APM, SDSS, NVSS) and the X-ray background (HEAO). Two combined analyses of the ISW effect with many galaxy samples have been undertaken by Ho et al. (2008) and Giannantonio et al. (2008) with a total detection significance of 4σ. In the matter+curvature case, in combination with CMB power spectrum, Ho et al. found $\Omega_m = 0.26^{+0.12}_{-0.07}$ and $\Omega_K = -0.01 \pm 0.02$. These are not the tightest constraints but are a welcome confirmation of the standard cosmology. Further improvement of these errors is difficult because the chance superposition of the background CMB fluctuations with the galaxy distribution introduces an unavoidable level of noise. Thus ISW will never be the best way to constrain standard cosmological parameters, but it may be a strong discriminator of non-standard explanations for the accelerating universe.

5 Bispectrum/non-Gaussianity

Earlier we mentioned β as a way to measure the bias so that the galaxy power spectrum can be converted to a matter power spectrum. Another way is to directly use the nonlinear evolution. This method pushes into regimes where galaxy formation physics may intervene, and hence is a very useful consistency check on the standard galaxy biasing picture. That is to use the non-Gaussian correlation between three Fourier modes, or "bispectrum" of the galaxies. We will only sketch the method here, details can be found in the application to 2dF by Verde et al. (2002) MNRAS 335, 432.

The density field today can be expanded in powers of the primordial density field via perturbation theory:

$$\delta_m = \delta^{(1)} + \delta^{(2)} + \delta^{(3)} + ..., \tag{40}$$

where

$$\delta^{(1)}(\mathbf{k}) = \frac{2}{5} D(a) T(k) \frac{k^2}{\Omega_m H_0^2} \zeta(\mathbf{k}) \tag{41}$$

is the linear theory term that we have calculated. The second-order term is generically written as

$$\delta^{(2)}(\mathbf{k}) = \int \frac{d^3\mathbf{k}'}{(2\pi)^3} J(\mathbf{k}', \mathbf{k} - \mathbf{k}') \delta^{(1)}(\mathbf{k}') \delta^{(1)}(\mathbf{k} - \mathbf{k}'), \tag{42}$$

where the coupling coefficient J is (after a long calculation):

$$J(\mathbf{k}_1, \mathbf{k}_2) = \frac{5}{7} + \frac{2}{7} (\hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2)^2 + \frac{1}{2} (\hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2) \left(\frac{k_1}{k_2} + \frac{k_2}{k_1} \right). \tag{43}$$

Then to second order in perturbation theory the density field exhibits a *bispectrum* or 3-mode correlation function:

$$\langle \delta_m(\mathbf{k}_1) \delta_m(\mathbf{k}_2) \delta_m(\mathbf{k}_3) \rangle = (2\pi)^3 B(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3), \tag{44}$$

where

$$B(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = 2J(\mathbf{k}_1, \mathbf{k}_2) P_m(k_1) P_m(k_2) + \text{permutations}. \tag{45}$$

We can't observe the matter bispectrum directly, but we can observe the galaxy bispectrum. To second order, the galaxy overdensity must be expanded to second order in the matter density:

$$\delta_g(\mathbf{x}) = 1 + b_1\delta_m(\mathbf{x}) + \frac{1}{2}b_2\delta_m(\mathbf{x})^2 + ... \tag{46}$$

The bispectrum of the galaxies can then be computed to get:

$$B_g(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = \left[2b_1^3 J(\mathbf{k}_1, \mathbf{k}_2) + b_1^2 b_2\right] P_m(k_1)P_m(k_2) + \text{permutations}. \tag{47}$$

If we recall that the matter power spectrum is $P_g(k)/b_1^2$, then we can substitute to get:

$$B_g(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = \left[\frac{2}{b_1} J(\mathbf{k}_1, \mathbf{k}_2) + \frac{b_2}{b_1^2}\right] P_g(k_1)P_g(k_2) + \text{permutations}. \tag{48}$$

The bispectrum is rich in information because we can measure it for any triangle of $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$. By using the configuration dependence of J, we can separately measure $2/b_1$ and b_2/b_1^2, so the large-scale bias of the galaxies can be measured from the bispectrum, and we get b_2 as a bonus. Verde et al. (2002) did this with the 2dF galaxies and found $b = 1.04 \pm 0.11$. By combining this with the measurement of β from 2dF, they derived $\Omega_m = 0.27 \pm 0.06$, entirely internal to 2dF (no reliance on CMB!)

6 Weak lensing

A final way to probe the matter distribution is with weak gravitational lensing. This is an extremely powerful method in principle, and quite hard in practice due to systematic effects. Its power lies in the fact that lensing is sensitive directly to the gravitational potential and hence the matter distribution. Its difficulty lies in that it is a small effect.

Gravitational deflection of light. We first wish to consider what happens to the images of distant galaxies if we look in the sky in direction $\hat{\mathbf{n}}$ through an inhomogeneous universe.

Imagine a light ray traveling in direction $\hat{\mathbf{p}}$. The geodesic equation allows one to solve for the deflection of $\hat{\mathbf{p}}$ in the presence of a gravitational potential; one finds:

$$\dot{\hat{\mathbf{p}}} = -2\nabla_\perp\Phi, \tag{49}$$

where $\nabla_\perp$ is the gradient operator perpendicular to $\hat{\mathbf{p}}$. The light ray that we see in direction $\hat{\mathbf{n}}$ has direction of propagation $\hat{\mathbf{p}} = -\hat{\mathbf{n}}$ today. If we integrate its trajectory backward, we find:

$$\hat{\mathbf{p}}(\eta_0 - r) = -\hat{\mathbf{n}} + 2\int \nabla_\perp\Phi(\mathbf{x})\, dr, \tag{50}$$

where $\mathbf{x}$ is the position of the photon at some earlier time. We will work in linear theory (Born approximation) so that we may approximate $\mathbf{x} = r\hat{\mathbf{n}}$. Also the perpendicular gradient is $\mathbf{D}/r$, where $\mathbf{D}$ is the angular covariant derivative on the unit sphere. Thus:

$$\hat{\mathbf{p}}(\eta_0 - r) = -\hat{\mathbf{n}} + 2\mathbf{D} \int \Phi(r\hat{\mathbf{n}}) \, \frac{dr}{r}. \tag{51}$$

The comoving position $\mathbf{x}$ of the photon at some earlier time is obtained by integrating $\hat{\mathbf{p}}$:

$$\begin{aligned} \mathbf{x}(\eta_0 - r) &= -\int \hat{\mathbf{p}}(\eta_0 - r') \, dr' \\ &= r\hat{\mathbf{n}} - 2\mathbf{D} \int_0^r dr' \int_0^{r'} \Phi(r''\hat{\mathbf{n}}) \, \frac{dr''}{r''} \\ &= r\hat{\mathbf{n}} - 2\mathbf{D} \int_0^r dr'' \frac{r - r''}{r''} \Phi(r''\hat{\mathbf{n}}) \, dr''. \end{aligned} \tag{52}$$

We define the lensing strength to be:

$$g(r''|r) = 2\frac{r''(r - r'')}{r}, \tag{53}$$

so that:

$$\mathbf{x}(\eta_0 - r) = r \left[\hat{\mathbf{n}} - \mathbf{D} \int_0^r g(r''|r)\Phi(r''\hat{\mathbf{n}}) \, \frac{dr''}{r''^2} \right]. \tag{54}$$

The second term is called the *deflection angle* $\mathbf{d}$:

$$\mathbf{d} = -\mathbf{D} \int_0^r g(r''|r)\Phi(r''\hat{\mathbf{n}}) \, \frac{dr''}{r''^2}, \tag{55}$$

and it is the angular gradient of the *lensing potential* ψ:

$$\psi = -\int_0^r g(r''|r)\Phi(r''\hat{\mathbf{n}}) \, \frac{dr''}{r''^2}. \tag{56}$$

The observed position of a galaxy $\hat{\mathbf{n}}$ and distance r is related to its true position $\hat{\mathbf{n}}^S$ via:

$$\hat{\mathbf{n}}^S = \hat{\mathbf{n}} + \mathbf{d}. \tag{57}$$

Usually $\hat{\mathbf{n}}^S$ and $\hat{\mathbf{n}}$ are called the *source-plane* and *image-plane* positions, respectively.

The lensing potential is positive in overdense regions (negative Φ) and negative in underdense regions. This makes sense: the deflection angle is toward overdensities.

Image distortions. Neither the lensing potential or the deflection angle is observable because we don't know the true positions of the galaxies we observe. We can start to examine observable quantities by taking derivatives of ψ. The key idea is that galaxies, on average, are round in the source plane: some are elongated in one direction, others in another, but on average they are round. To make this explicit, let's write the Jacobian for the transformation from source to image plane:

$$\mathrm{Jac}_{ij} = \frac{\partial n_i}{\partial n_j^S} = \delta_{ij} - d_{i,j} = \delta_{ij} - \psi_{,ij}. \tag{58}$$

Often the distortion matrix $d_{i,j}$ is decomposed into a diagonal part and a 2×2 traceless-symmetric tensor:

$$A_{ij} \equiv -\psi_{,ij} = -\kappa\delta_{ij} - \begin{pmatrix} \gamma_Q & \gamma_U \\ \gamma_U & -\gamma_Q \end{pmatrix}. \tag{59}$$

In this matrix, κ corresponds to an overall magnification of the image; γ_Q is a shear, i.e. stretch along the N-S axis and a compression along the E-W axis; and γ_U is a shear that stretches along the NE-SW axis and compresses along NW-SE. All three of these are measurable if one averages sizes or shapes over an ensemble of galaxies. The usual preference is for shapes because the size distribution of galaxies is wider than the shape distribution, so the γ measurement is much less noisy.

There is a large literature on ways to estimate γ from galaxy shapes; we won't get into them here but this is a topic of current research.

Shear power spectrum. Here κ forms a scalar field on the sphere, and $\gamma_{Q,U}$ form a tensor (just like polarization components). Hence κ has a power spectrum, and γ has E- and B-type power spectra. Because it is derived by differentiating a scalar, γ has no B-type components. In fact, one can show that in the multipole decomposition:

$$\kappa_{lm} = l(l+1)\psi_{lm}; \quad \gamma^E_{lm} = \sqrt{(l-1)l(l+1)(l+2)}\,\psi_{lm}; \quad \gamma^B_{lm} = 0. \tag{60}$$

The E-type power spectrum of γ can be computed from first principles given a set of cosmological parameters; it is an integral over the matter power spectrum and hence is proportional to σ_8^2 in linear perturbation theory. It thus provides yet another way to measure σ_8 which cannot be obtained directly from the galaxy power spectrum, independent of the bispectrum or the redshift-space distortions. But the most tempting way to use the E-type power spectrum is to directly use the E-type power spectrum to fit cosmological parameters. This way one is not sensitive to details of the galaxy formation process. Another advantage is that one can use simulations to follow the distribution of matter into the nonlinear regime where one does not know how galaxies will behave. This has been done, e.g. the CFHT Legacy Survey finds $\sigma_8(\Omega_m/0.25)^{0.64} = 0.785 \pm 0.043$ (Fu et al 2007).

The shear power spectrum is extremely useful, however it is subject to a number of systematic errors:

- *Point-spread function*: Galaxies at cosmological distances $z \geq 0.5$ are typically of order an arcsecond in size. This is the same order of magnitude as the smearing of light by the atmosphere. Most telescopes have optical distortions or tracking errors that are not much smaller than this. These effects can coherently distort the shapes of galaxies, and lead to a false shear signal.

- *Redshift distributions*: The shear power spectrum is a strong function of the redshifts of the galaxies. In order to measure shear one needs many more galaxies than one can obtain spectroscopic redshifts. Therefore one must obtain the redshift distribution, or better use photometric redshifts where one constructs a model spectrum based on stellar age, dust extinction, redshift, etc. and fits it to observed fluxes in several wavebands. Model deviations are sources of systematic error and must be tested against spectroscopic redshifts.

- *Intrinsic alignments*: The simple picture of shear outlined above assumes that galaxy shapes are random and average down as $1/\sqrt{N}$. Unfortunately some galaxies exhibit correlations of their intrinsic shapes that contaminate the lensing effect. The intrinsic shapes can also be correlated with the density field, which causes an additional type of systematic error. Current shear power spectrum measurements use small samples of spectroscopic-z galaxies to estimate and remove the effect. In the future it may be possible to eliminate it by using its different redshift dependence.

Galaxy-shear correlation. Another way to use weak lensing that is much less sensitive to these problems is to cross-correlate galaxies and shear. The correlation is proportional to $b\sigma_8^2$, whereas the galaxy power spectrum is proportional to $b^2\sigma_8^2$. The combination allows determination of σ_8.